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THE SCATTERING OF SOUND BY A SIMPLE SHEAR LAYER

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CONTENTS

A fixed line source, oscillating harmonically in time, produces sound waves which fall on a two dimensional shear layer in which the velocity increases linearly over a finite distance and then remains constant. The linearized theory of sound allows a multiplicity of solutions. The ambiguity is resolved by an application of the principle of causality. As a result it is found that, for Strouhal numbers below a certain critical level, Helmholtz instability is evident but not if the Strouhal number is above critical. The instability wave fans out from a negligibly small region as the Strouhal number drops from critical until it occupies a wedge of 45° when the layer simplifies to a vortex sheet. The limit is the same as that derived by direct analysis of the vortex sheet but no ultra-distributions are necessary if the layer is not infinitesimally thin.

Various other aspects of thin and thick layers are also discussed.

1. Introduction

The problem of the interaction of sound waves and jets has attracted much attention in recent years. The work has been stimulated by a desire for fundamental understanding of the process in order to quieten the noise generated by flows and machinery.

The simplest model that can be contemplated of the situation is a plane vortex sheet, separating a still medium from one moving with uniform velocity and illuminated by acoustic radiation from a line source. This model was discussed by Jones & Morgan (1972) (where earlier references will be found) when the source is fixed and by Jones (1974) when the source is moving. In both cases difficulties arose in ensuring that there was no sound field before the source was switched on, being resolved ultimately by working with rather abstruse entities known as ultradistributions. The physical interpretation of ultradistributions is not obvious and so doubt was cast on the adequacy of the model.

The model next in order of complexity is a shear layer in which the velocity increases linearly from 0 to the value U over the distance h and then remains constant. The layer is irradiated by sound from a line source. This model is considered in this paper on the assumption that the source is varying harmonically in time but fixed in space. Apart from its relevance to the noise of jets it has its own intrinsic interest as conveying some idea of the propagation of sound in a shearing wind.

One of the aims is to determine and evaluate the solution unequivocally so that a firm comparison between the cases $h = 0$ and $h \neq 0$ can be made. The case $h = 0$ has already displayed the difficulties of interpretation which can arise if due mathematical attention is not paid so no effort has been spared subsequently to guarantee that the analysis is as rigorous as can be.

The trouble with the investigation of the vortex sheet stemmed from the presence of a certain non-real pole in the complex plane. A careful examination of the poles for the general shear layer was therefore essential. After a lengthy elaborate study it was discovered that there was an infinite number of poles some of which were on the real axis. The multiplicity of solutions for the shear layer is thereby far worse than that for the vortex sheet. However, it was found that as the Strouhal number kh/M (k being the wavenumber of the source and M the Mach number associated with U) reduced two of the real zeros came into coincidence and shifted off the real axis. The observation turned out to be crucial in resolving some of the non-uniqueness especially since most of the poles ultimately gave an insignificant contribution to the field. The final arbiter, however, which disposed of all questions of uniqueness is the principle of causality. Fortunately, in this case, the conditions of causality can be complied with by conventional functions and ultradistributions do not have to be introduced.

The coalescence of the two real zeros referred to is related to the onset of Helmholtz instability in the shear layer and happens at a critical Strouhal number. If the Strouhal number is above critical there is no instability wave. As the Strouhal number moves below critical an instability wave first appears in the immediate neighbourhood of the downstream end of the layer and then gradually fans out eventually filling a wedge of 45° in the limit of vanishing Strouhal number.

The unstable behaviour of a layer was examined many years ago by Rayleigh (1880) but he confined his investigation to the instability of an incompressible jet under disturbances of its boundary. In our problem compressibility is an essential feature and there is an interaction between the acoustic excitation and fluid flow. Thus the phenomena are substantially more complicated than those encountered by Rayleigh. In addition, the problem is not one of determining which modes of a jet grow or decay but of finding the reaction to a specific sonic excitation. Problems similar to that of Rayleigh for compressible flow have been studied by Lilley (1974) and the compressible form of the Orr–Sommerfeld equation involved is related to the differential equation it will be necessary to analyse subsequently.

Instability waves may or may not be induced by an acoustic wave falling on a shear layer. When an instability wave is created it grows exponentially but other waves do not so the instability wave dominates in the region it occupies. That region depends upon the Strouhal number as already explained and may be non-existent. Since the critical Strouhal number is about $\frac{1}{3}$ there will be many practical situations in which the Strouhal number will not be small enough for an instability wave to be generated. For example, if the velocity transition across the layer corresponds to a

Mach number of 0.3 the layer thickness must be less than $\frac{1}{2}$ cm at a wavelength of 30 cm for instability to occur. In air, this would imply a velocity transition of 100 m s^{-1} and an acoustic frequency of 1100 Hz. At 200 Hz or a wavelength of 1.65 m, an upper bound for the thickness would be 3 cm for the same velocity change. On the other hand, a velocity change of 10 m s^{-1} excited at 1100 Hz would require a layer thinner than $\frac{1}{2}$ mm for instability to be observed.

There is a further aspect which may militate against the appearance of an instability wave. The instability grows from downstream so that in a real jet it has a tendency to try to originate in a region where the shear layer is widening. The effective Strouhal number is therefore increasing and consequently acting against the establishment of an instability wave. Nevertheless, it is important from a theoretical point of view to compare the very thin layer with the vortex sheet.

In fact the limit of the infinitesimally thin layer gives the same picture as the direct analysis of the vortex sheet and thus provides a confirmation of the associated theory. This is a satisfactory aspect of the work.

It should be emphasized that if the harmonic problem is considered in isolation there is no *a priori* reason for selecting the particular solution with its attached instability wave that was finally settled on. There are no criteria for the harmonic problem, if causality be set aside, for preventing the addition of arbitrary multiples of the instability wave and the waves from other poles to any given solution; they all satisfy the governing equations. Thus the principle of causality is an essential weapon in arriving at the correct solution of acoustic scattering by shear layers.

The problem of the harmonic source is formulated in section 2 and a mathematical solution determined. To find the solution it is necessary to solve a certain differential equation. In order not to interrupt the main text the properties of this differential equation are discussed in appendix A; equations in this appendix are identified by the letter A so that, for example, (A 1) means equation (1) of appendix A.

The field in the still medium at large distances from the origin is examined in § 3. This necessitates a knowledge of the poles of the integrand, a matter which is investigated in appendix B. Plausible arguments are advanced for electing one of the several solutions as the candidate appropriate to the problem. The consequent behaviour of the instability wave has already been described and the evaluation of other waves is also carried out.

The field produced on the far side of the shear layer is calculated in § 4. In § 5 the modifications which occur when the source moves off to infinity and the incident radiation becomes a plane wave are derived, and the relation between the reflexion and transmission coefficients elucidated. The special cases when the layer is thin or thick are discussed in §§ 6 and 7 respectively. If the layer is thin, i.e. the exciting wavelength is much greater than the thickness, the solution approximates to that for the vortex sheet, the approximation improving as $h \to 0$. The thick layer, which is many wavelengths across, either does not reflect much energy and most of the energy is transmitted or the wave is totally reflected while the transmitted wave is evanescent. The solution for the thick layer is compared in § 8 with that supplied by ray theory and satisfactory agreement found. In addition, it is suggested that observations of energy perpendicular to the layer may have some difficulty in recognizing the presence of the layer.

The content of § 9 is devoted to the implications of applying the principle of causality. In contrast to the case of the vortex sheet, it is brought to light that a contour exists which provides a unique causal solution which can be expressed in conventional functions and does not have to be synthesized from ultradistributions. By analytic continuation, it is confirmed that the solution adopted in § 3 complies with the principle of causality and is the only one so to do. A final paragraph concerns a simple example so that one may perceive why ultradistributions can be steered clear of when $h \neq 0$ but not when $h = 0$.

The last section, 10, very briefly summarizes the main results of the preceding sections in qualitative fashion.

2. Formulation

The problem to be considered is that of the propagation of two dimensional sound waves in the presence of a simple shear layer. In the undisturbed state the fluid will be assumed to be flowing parallel to the x-axis with velocity independent of x though varying with y, the coordinate perpendicular to x. The undisturbed medium is taken to be at rest in $y < 0$. The shear layer occupies the region $0 < y < h$ and in it the only component of velocity is $U(y)$ parallel to the x-axis· A simple model is chosen in which $U(y)$ increases linearly from zero at $y = 0$ to $U(h)$ at $y = h$ so that

$$U(y) = U(h)\, y/h \quad (0 < y < h).$$

Above the level $y = h$, $U(y)$ has the constant value $U(h)$. The constant pressure and density will be denoted by p_0 and ρ_0 respectively. Also the fluid will be assumed to be inviscid and the sound speed a taken as constant.

Sound waves are now superimposed on the flow. These will perturb the pressure to $p_0(1+p)$ and the density to $\rho_0(1+\rho)$. The velocity will have components $U(y)+u_x$ and u_y parallel to the x- and y-axes respectively. It will be assumed that the perturbations are sufficiently small for linearized theory to be employed. Then the governing equations are

$$\frac{\partial u_x}{\partial t} + U(y)\frac{\partial u_x}{\partial x} + u_y U' + \frac{p_0}{\rho_0}\frac{\partial p}{\partial x} = 0, \tag{1}$$

$$\frac{\partial u_y}{\partial t} + U(y)\frac{\partial u_y}{\partial x} + \frac{p_0}{\rho_0}\frac{\partial p}{\partial y} = 0, \tag{2}$$

$$\frac{\partial \rho}{\partial t} + U(y)\frac{\partial \rho}{\partial x} + \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0, \tag{3}$$

where t is time and the effects of thermal conductivity and gravity are neglected. For the moment, the source of sound has been omitted.

Let the time variation of the acoustic wave be harmonic, varying according to the factor $e^{i\omega t}$ which will be suppressed subsequently. Then, if $dp = a^2\rho_0 d\rho/p_0$, where a is the constant speed of sound in the undisturbed flow, equations (1)–(3) reduce to

$$\left\{U(y)\frac{\partial}{\partial x} + i\omega\right\} u_x + U' u_y + a^2\frac{\partial \rho}{\partial x} = 0, \tag{4}$$

$$\left\{U(y)\frac{\partial}{\partial x} + i\omega\right\} u_y + a^2\frac{\partial \rho}{\partial y} = 0, \tag{5}$$

$$\left\{U(y)\frac{\partial}{\partial x} + i\omega\right\} \rho + \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0. \tag{6}$$

From (4)–(6) follows

$$\left\{U(y)\frac{\partial}{\partial x} + i\omega\right\}\left\{U(y)\frac{\partial}{\partial x} + i\omega\right\}\rho - a^2\left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2}\right) - 2U'\frac{\partial u_y}{\partial x} = 0. \tag{7}$$

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When there is no shear flow so that $U' = 0$, the acoustic source customarily appears on the right hand side of (7). It will therefore be added to (7) in this case. In effect, it can be regarded as coming from the continuity equation (3) and representing the creation or destruction of mass. Accordingly, (7) is replaced by

$$\left\{U(y)\frac{\partial}{\partial x}+\mathrm{i}\omega\right\}^2\rho-a^2\left(\frac{\partial^2\rho}{\partial x^2}+\frac{\partial^2\rho}{\partial y^2}\right)-2U'\frac{\partial u_y}{\partial x}=\delta(x)\,\delta(y-y_0), \tag{8}$$

where $\delta(x)$ is the usual Dirac delta function. The source of sound has been placed at $(0, y_0)$ and, in the following, y_0 will be chosen as negative so that the source lies in the still medium.

Coupled with (8) are (4) and (5); these are the equations which have to be solved in our model. We seek solutions of the form

$$\rho=\frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{\rho}\,\mathrm{e}^{-\mathrm{i}kux}\,\mathrm{d}u$$

with similar expressions for the velocity components, where k is the wave number ω/a. The strict nature of the contour of integration will be decided later. In the meantime we proceed formally. Equations (4), (5) and (8) give

$$\tilde{u}_y=a^2[\mathrm{i}k\{uU(y)-a\}]^{-1}\,\mathrm{d}\tilde{\rho}/\mathrm{d}y, \tag{9}$$

$$\mathrm{i}k\{uU(y)-a\}\,\tilde{u}_x=-\mathrm{i}kua^2\tilde{\rho}+a^2U'(y)\,[\mathrm{i}k\{uU(y)-a\}]^{-1}\,\mathrm{d}\tilde{\rho}/\mathrm{d}y, \tag{10}$$

$$-k^2\{uU(y)-a\}^2\tilde{\rho}-a^2(-k^2u^2\tilde{\rho}+\mathrm{d}^2\tilde{\rho}/\mathrm{d}y^2)+2U'a^2u\{uU(y)-a\}^{-1}\,\mathrm{d}\tilde{\rho}/\mathrm{d}y=k\,\delta(y-y_0). \tag{11}$$

In the region $y < 0$ where $U \equiv 0$, (11) reduces to

$$\mathrm{d}^2\tilde{\rho}/\mathrm{d}y^2+k^2v^2\tilde{\rho}=-k\delta(y-y_0)/a^2, \tag{12}$$

where $v^2 = 1-u^2$, while in $y > h$ where $U \equiv U(h)$, (11) becomes

$$\mathrm{d}^2\tilde{\rho}/\mathrm{d}y^2+k^2w^2\tilde{\rho}=0, \tag{13}$$

where $w^2 = (1-Mu)^2-u^2$, with $M = U(h)/a$ the Mach number associated with the stream above $y = h$.

So far the fact that U varies linearly in $0 < y < h$ has not been used. Make the substitution $Y = s-kuy$ where $s = kh/M$ is the Strouhal number of the flow in $y > h$. Then, in $0 < y < h$, (11) is transformed to

$$\mathrm{d}^2\tilde{\rho}/\mathrm{d}Y^2-(2/Y)\,\mathrm{d}\tilde{\rho}/\mathrm{d}Y-(1-Y^2/\tau^2)\,\tilde{\rho}=0, \tag{14}$$

where $\tau = us$.

Differential equations related to (14) have been encountered by Goldstein (1974) and Lilley (1974) though not for the ranges of parameters involved here.

Before proceeding to the solution of (12)–(14) it is necessary to indicate the meaning to be attached to v and w for complex values of u. The branch lines for v will be drawn along the real u-axis from $-\infty$ to -1 and from 1 to ∞. Then v will be required to be positive for $-1 < u < 1$; it will be negative imaginary for u just above the branch line from 1 to ∞ and also when u is just below the branch line from $-\infty$ to -1.

As regards w, it will be assumed from now on that $M < 1$ so that the stream is moving subsonically in $y > h$. In view of the linear variation of $U(y)$ the flow is, in fact, subsonic for all y. The branch lines of w are drawn along the real axis from $-\infty$ to $-1/(1-M)$ and from $1/(1+M)$ to ∞. On the real axis between the branch points w will be taken to be positive. Just above and below its branch lines it will have similar behaviour to that of v. In the following we shall have in mind that u is real, approaching the real axis from above in $u > 0$ and from below in $u < 0$.

If there is slight absorption present so that k has a negative imaginary part it is to be expected that the sound waves will decay as $|y| \to \infty$. Therefore, in view of the choice of branches for v and w, the appropriate solutions of (12) and (13) are

$$\begin{aligned}\tilde{\rho} &= A\exp[\mathrm{i}kvy] + (1/2\mathrm{i}a^2v)\exp[-\mathrm{i}kv|y-y_0|] \quad (y < 0)\\ &= B\exp[-\mathrm{i}kwy] \quad (y > h).\end{aligned}$$

For $0 < y < h$, (14) must be solved. The two independent solutions of (14) are examined in appendix A and denoted by f, g. The pair must be retained in $0 < y < h$ since they are both finite in the region and so

$$\tilde{\rho} = Cf(Y) + Dg(Y) \quad (0 < y < h).$$

The continuity of ρ and u_y across the interfaces $y = 0$ and $y = h$ implies that

$$A + (1/2\mathrm{i}a^2v)\,\mathrm{e}^{\mathrm{i}kvy_0} = Cf(s) + Dg(s),$$

$$\mathrm{i}kv\{A - (1/2\mathrm{i}a^2v)\,\mathrm{e}^{\mathrm{i}kvy_0}\} = -ku\{Cf'(s) + Dg'(s)\},$$

$$B\,\mathrm{e}^{-\mathrm{i}kwh} = Cf(\xi) + Dg(\xi),$$

$$-\mathrm{i}wB\,\mathrm{e}^{-\mathrm{i}kwh} = -u\{Cf'(\xi) + Dg'(\xi)\},$$

where $\xi = s - kuh$. Therefore

$$\Delta A = \frac{1}{2\mathrm{i}a^2v}\exp[\mathrm{i}kvy_0]\,[\mathrm{i}v\{\mu f(s) - \lambda g(s)\} - u\{\mu f'(s) - \lambda g'(s)\}], \tag{15}$$

$$\Delta B = -(3\xi^2u/a^2)\exp[\mathrm{i}k(vy_0 + wh)], \tag{16}$$

$$\Delta C = (\mu/a^2)\,\mathrm{e}^{\mathrm{i}kvy_0}, \tag{17}$$

$$\Delta D = -(\lambda/a^2)\,\mathrm{e}^{\mathrm{i}kvy_0}, \tag{18}$$

where

$$\lambda = uf'(\xi) - \mathrm{i}wf(\xi), \quad \mu = ug'(\xi) - \mathrm{i}wg(\xi), \tag{19}$$

$$\Delta = \mu\{uf'(s) + \mathrm{i}vf(s)\} - \lambda\{ug'(s) + \mathrm{i}vg(s)\} \tag{20}$$

and, in the derivation of B, the Wronskian relation (A 8) has been employed.

The formulae for the disturbance in $y < 0$ are now

$$\rho = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left\{A\exp[\mathrm{i}kvy] + \frac{1}{2\mathrm{i}a^2v}\exp[-\mathrm{i}kv|y-y_0|]\right\}\exp[-\mathrm{i}kux]\,\mathrm{d}u, \tag{21}$$

$$u_x = \frac{a}{2\pi}\int_{-\infty}^{\infty}\left\{A\exp[\mathrm{i}kvy] + \frac{1}{2\mathrm{i}a^2v}\exp[-\mathrm{i}kv|y-y_0|]\right\}\exp[-\mathrm{i}kux]\,u\,\mathrm{d}u, \tag{22}$$

$$u_y = -\frac{a}{2\pi}\int_{-\infty}^{\infty}\left\{Av\exp[\mathrm{i}kvy] + \frac{\operatorname{sgn}(y-y_0)}{2\mathrm{i}a^2}\exp[-\mathrm{i}kv|y-y_0|]\right\}\exp[-\mathrm{i}kux]\,\mathrm{d}u, \tag{23}$$

where A is given by (15). In $y > h$,

$$\rho = \frac{1}{2\pi}\int_{-\infty}^{\infty}B\exp[-\mathrm{i}kwy - \mathrm{i}kux]\,\mathrm{d}u, \tag{24}$$

$$u_x = \frac{a}{2\pi}\int_{-\infty}^{\infty}\frac{uB}{1-Mu}\exp[-\mathrm{i}kwy - \mathrm{i}kux]\,\mathrm{d}u, \tag{25}$$

$$u_y = \frac{a}{2\pi}\int_{-\infty}^{\infty}\frac{wB}{1-Mu}\exp[-\mathrm{i}kwy - \mathrm{i}kux]\,\mathrm{d}u, \tag{26}$$

while in $0 < y < h$

$$\rho = \frac{1}{2\pi}\int_{-\infty}^{\infty} \{Cf(Y) + Dg(Y)\} \exp[-ikux]\, du, \tag{27}$$

$$u_x = \frac{as}{2\pi}\int_{-\infty}^{\infty} [C\{Yf(Y) + f'(Y)\} + D\{Yg(Y) + g'(Y)\}] \exp[-ikux]\, u\, du/Y^2, \tag{28}$$

$$u_y = \frac{as}{2\pi i}\int_{-\infty}^{\infty} \{Cf'(Y) + Dg'(Y)\} \exp[-ikux]\, u\, du/Y. \tag{29}$$

The integrands in (21)–(29) may contain singularities at the branch points of v and w, at the zeros of Δ, at $u = 1/M$ and at $u = s/ky$. There may also be an essential singularity at $u = 0$ because τ vanishes there and f, g have an essential singularity at $\tau = 0$. To avoid these possible singularities the path of integration in the u-plane is chosen to be the real axis with indentations. The path passes below the branch lines in $u < 0$ and above the branch lines in $u > 0$. The indentations are below the singularities in $u < 0$ and above them in $u > 0$ but, for definiteness, the indentation at $u = 0$ is placed above. It will be found subsequently that the integrands do not, in fact, have any singularity at $u = 0$ so that the indentation there is not strictly necessary.

With this understanding the integrals in (21)–(29) will exist unless convergence at infinity fails. Now, as $u \to \infty$ just above the real axis,

$$\lambda \sim uf'(-kuh) - iwf(-kuh)$$
$$\sim uf'(kuh) + (1-M^2)^{\frac{1}{2}}\, uf(kuh)$$

from (A 4). To apply (A 23) and (A 24) note that $\tau = us$ and so ζ is effectively M. Therefore z of (A 16) is a positive constant. Taking advantage of (A 25) we obtain

$$\lambda \sim 3u\tau M(1-M^2)^{\frac{1}{4}} \exp[\tfrac{1}{2}\tau\{\arcsin M + M(1-M^2)^{\frac{1}{2}}\} - s(1-M^2)^{\frac{1}{2}}]. \tag{30}$$

Similarly, from (A 31), $\mu \sim \frac{1}{3}\lambda$. From (A 51)–(A 54),

$$\Delta \sim \lambda u\, e^{s}(2s-1). \tag{31}$$

Therefore the integrals involving A and B in (21)–(26) converge uniformly as $u \to \infty$. At first sight, the integrals in (27)–(29) appear to diverge because of the combinations $Yf + f'$. However, it is straightforward to check that the exponentially growing terms do, in fact, cancel one another by grouping f with g and employing (A 30), (A 35) or by substituting f_1 and g_1 for f and g via (A 36), (A 37) and using (A 47,) (A 48).

As $u \to -\infty$, just below the real axis

$$\lambda \sim uf'(kh|u|) + (1-M^2)^{\frac{1}{2}}\, uf(kh|u|)$$

and $\tau \to -\infty$. Replacing τ by $-\tau$ on account of (A 5), we find that (30) is still true with $|\tau|$ in place of τ. Again, the integrals converge at $-\infty$, uniformly with respect to x and y.

By virtue of the uniformity of convergence, ρ and the velocity are continuous everywhere, except possibly at $y = y_0$. Taking derivatives with respect to x or y does not destroy the uniformity of convergence. The terms not involving A in (21)–(23) provide a particular integral of (8). Thus we can see that (21)–(29) supplies a solution to our problem which satisfies all the conditions imposed as long as the conventions introduced for the path of integration are observed.

In the next section the implications of these formulae are investigated. Attention will be concentrated primarily on the density since there is no great difficulty in deriving from its properties the main features of the velocity.

3. The reflected wave

The second part of the integrand of (21) gives rise to an integral which can be evaluated in terms of Hankel functions. A similar evaluation can be carried out for part of Δ. The net result is that, in the region $y < 0$, (21) gives

$$\rho = \frac{\mathrm{i}}{4a^2}\{H_0^{(2)}(kr_1) - H_0^{(2)}(kr)\} + \frac{1}{2\pi a^2}\int_{-\infty}^{\infty}\frac{1}{\Delta}\{\mu f(s) - \lambda g(s)\}\exp\left[\mathrm{i}kv(y+y_0) - \mathrm{i}kux\right]\mathrm{d}u$$

where $H_0^{(2)}$ is the Hankel function of the second kind, $r = \{x^2 + (y-y_0)^2\}^{\frac{1}{2}}$ and $r_1 = \{x^2 + (y+y_0)^2\}^{\frac{1}{2}}$. The term involving r represents the incident wave produced by the source as if it were in a medium of infinite extent with $U \equiv 0$. The other terms furnish the acoustic field reflected by the shear layer and they will be denoted by ρ_{r}.

The reflected wave ρ_{r} consists of two constituents. The first, the Hankel function with argument kr_1, can be thought of as the wave reflected by a soft boundary (where $\rho = 0$) at $y = 0$. The second, comprising the integral, accounts for the presence of the shear layer and its modification of the simple soft boundary condition. If we write

$$\rho_{\mathrm{s}} = \frac{1}{2\pi a^2}\int_{-\infty}^{\infty}\frac{1}{\Delta}\{\mu f(s) - \lambda g(s)\}\exp\left[\mathrm{i}kv(y+y_0) - \mathrm{i}kux\right]\mathrm{d}u$$

so that $\rho_{\mathrm{r}} = (\mathrm{i}/4a^2)\,H_0^{(2)}(kr_1) + \rho_{\mathrm{s}}$, it is only when ρ_{s} is small compared with the Hankel function that it is reasonable to regard the shear layer as acting as a soft interface.

The evaluation of ρ_{s} depends crucially upon the zeros of Δ. The occurrence of even a single non-real zero of Δ can lead to substantial trouble in ensuring a causal solution in the time domain (see, for example, Jones & Morgan 1972). In effect, a non-real zero can generate a solution of (12)–(14) with zero right hand side. By adding this solution to (21)–(29) we can manufacture another answer fulfilling the conditions of our problem. Thus there is no uniqueness and the proper interpretation of formulae is hard. As will be seen, Δ does have non-real zeros and so a correct appreciation of their significance is a cardinal aspect of the investigation (see § 9).

Locating the zeros of Δ is an intricate and delicate matter; details are in appendix B, with a summary in § B 11. Briefly, Δ has a zero at the origin and infinite sequences of non-real zeros $u_1^+, u_2^+, \ldots, u_1^-, u_2^-, \ldots$ in the upper and lower half-planes respectively for all values of s. In addition, Δ has one real zero (in $u > 1/M$) for $s > \frac{1}{2}$ and two real zeros (both in $u > 1/M$) if $\frac{1}{2} > s > s_0$, where s_0 is a certain critical Strouhal number. If $s < s_0$, the two real zeros dissolve but a further non-real zero u_0 presents itself in the upper half-plane.

The zero of Δ at the origin is counterbalanced by one in the numerator of each of the integrands (21), (24) and (27) as can be verified from (A 11)–(A 14). Therefore, the zero of Δ at the origin will be ignored from now on. The other real zeros which may be encountered are circumvented by little detours above.

An estimate of ρ_{s} at points of observation well away from the origin will now be derived by the method of stationary phase. Let $x = r_1\cos\theta_1$, $y + y_0 = -r_1\sin\theta_1$ where $0 < \theta_1 < \pi$. Assume that $kr_1 \gg 1$. Then $u = \cos\theta_1$ is a point of stationary phase. Deform the contour of integration into the hyperbola $u = \cos(\theta_1 - \mathrm{i}\sigma)$ where σ is real and runs from $-\infty$ to ∞. No branch line is met in the deformation if $\theta_1 > \gamma$, where γ is the angle between 0 and $\frac{1}{2}\pi$ such that $\cos\gamma = 1/(M+1)$. If $\theta_1 < \gamma$, the hyperbolic path must be supplemented by a loop round the branch point at $u = 1/(M+1)$.

In the deformation poles due to the zeros of Δ may be passed over. To assess their contribution let $u_0 = \cos(\phi_0 - i\sigma_0)$ and $u_j^{\pm} = \cos(\phi_j^{\pm} - i\sigma_j^{\pm})$ where $0 < \phi_0 < \pi$ and $0 < \phi_j^{\pm} < \pi$. Then u_j^{+} is passed over positively if $\theta_1 > \phi_j^{+}$ and u_j^{-} negatively if $\theta_1 < \phi_j^{-}$. Hence, as $kr_1 \to \infty$,

$$\begin{aligned}\rho_s \sim \frac{1}{2a^2}\frac{R_0}{(2\pi kr_1)^{\frac{1}{2}}}\exp[-ikr_1 - \tfrac{1}{4}\pi i] + \frac{i}{a^2}\sum_{j=0}\frac{1}{\Delta'(u_j^{+})}H(\theta_1 - \phi_j^{+})\{\mu_j^{+}f(s) - \lambda_j^{+}g(s)\}\exp[-ika_j] \\ - \frac{i}{a^2}\sum_{j=1}\frac{1}{\Delta'(u_j^{-})}H(\phi_j^{-} - \theta_1)\{\mu_j^{-}f(s) - \lambda_j^{-}g(s)\}\exp[-ikb_j] \\ + H(\gamma - \theta_1)\frac{9s^4\cos^4\gamma}{(2\pi)^{\frac{1}{2}}a^2b^2}\left\{\frac{\sin\gamma}{kr_1\sin(\gamma - \theta_1)}\right\}^{\frac{3}{2}}\exp[\tfrac{1}{4}\pi i - ikr_1\cos(\theta_1 - \gamma)],\end{aligned} \tag{32}$$

where $R_0 = 2i\sin\theta_1\{\mu f(s) - \lambda g(s)\}/\Delta$ evaluated at $u = \cos\theta_1$, $\lambda_j^{\pm}$ and $\mu_j^{\pm}$ are the values of λ and μ respectively at $u = u_j^{\pm}$, $f(s)$ means $f(s, su_j^{\pm})$ in a typical term of the series, $\Delta'(u) = d\Delta/du$, $H(x)$ is the Heaviside unit function which is unity for $x > 0$, $\frac{1}{2}$ for $x = 0$ and zero for $x < 0$, $a_j = u_j^{+}x - v_j^{+}(y+y_0)$, $b_j = u_j^{-}x - v_j^{-}(y+y_0)$, $v_j^{\pm} = (1-u_j^{\pm 2})^{\frac{1}{2}}$ and

$$b = [g'(\xi)\{uf'(s) + ivf(s)\} - f'(\xi)\{ug'(s) + ivg(s)\}]_{u=\cos\gamma}.$$

For temporary convenience the contribution from the pole $u = u_0$ has been included with the series for u_j^{+}. It is understood that in R_0, w stands for $-i|w|$ when $\theta_1 < \gamma$.

The formula (32) is not valid if θ_1 is near $\phi_j^{\pm}$ or γ. Transition formulae are required in these regions. They may be derived as in Jones (1964) for θ_1 near ϕ_j and as in Jones & Morgan (1972) for θ_1 near γ. The thorough discussion in Jones & Morgan (1972) of the transition that takes place as variations in θ_1 move the contour through a pole indicates that it provides a negligible correction to the term involving R_0 so long as the pole does not have a tiny imaginary part. The only pole which approaches the real axis is u_0 so that all other poles may now be dropped from (32).

Expand the Hankel function in ρ_r asymptotically. Then, as $kr_1 \to \infty$,

$$\begin{aligned}\rho_r \sim \frac{1}{2a^2}\frac{R}{(2\pi kr_1)^{\frac{1}{2}}}\exp[-ikr_1 - \tfrac{1}{4}\pi i] \\ + H(\gamma - \theta_1)\frac{9s^4\cos^4\gamma}{(2\pi)^{\frac{1}{2}}a^2b^2}\left\{\frac{\sin\gamma}{kr_1\sin(\gamma - \theta_1)}\right\}^{\frac{3}{2}}\exp[\tfrac{1}{4}\pi i - ikr_1\cos(\theta_1 - \gamma)]\end{aligned} \tag{33}$$

when $s > s_0$; here $R = R_0 - 1$. If $s < s_0$

$$\begin{aligned}\rho_r \sim \frac{1}{2a^2}\frac{R}{(2\pi kr_1)^{\frac{1}{2}}}\exp[-ikr_1 - \tfrac{1}{4}\pi i] \\ + H(\gamma - \theta_1)\frac{9s^4\cos^4\gamma}{(2\pi)^{\frac{1}{2}}a^2b^2}\left\{\frac{\sin\gamma}{kr_1\sin(\gamma - \theta_1)}\right\}^{\frac{3}{2}}\exp[\tfrac{1}{4}\pi i - ikr_1\cos(\theta_1 - \gamma)] \\ + \frac{i}{a^2}\frac{H(\theta_1 - \phi_0)}{\Delta'(u_0)}\{\mu_0 f(s) - \lambda_0 g(s)\}\exp[-ika_0].\end{aligned} \tag{34}$$

Neither (33) nor (34) holds for θ_1 near γ.

The first terms of (33) and (34) can be regarded as representing a wave specularly reflected by the shear layer, with R being counted as a reflexion coefficient which depends upon the angle of observation θ_1 and the Strouhal number s. The remaining term of (33), and its counterpart in (34), is of a lower order and corresponds to a bow wave, generated by the primary wave exciting a disturbance in the shear layer which travels faster downstream than the primary wave moves along $y = 0$.

If $\theta_1 < \gamma$ the bow wave is present and the convention on the meaning of w in R_0 leads to $|R| = 1$. Thus the bow wave arises when the phenomenon of total reflexion occurs.

If $\theta_1 > \gamma$,

$$1-|R|^2 = 36vwu^2s^2\xi^2/|\Delta|^2$$

evaluated at $u = \cos\theta_1$. Consequently, in those regions where the bow wave is absent, the reflexion coefficient satisfies $|R| < 1$ and the energy is no longer totally reflected.

As s passes through s_0, there is a discontinuity between (33) and (34) in the region $\theta_1 > \phi_0$. The discontinuity is initiated by u_0 leaving the real axis. It occupies practically the whole of the space below the interface except for a small neighbourhood of the downstream direction where it takes the form of a wave travelling practically undamped parallel to the stream but exponentially attenuated perpendicular to the flow. As s is reduced the damping becomes heavy in both directions and the last term in (34) can be neglected.

The question is whether such a discontinuous change should be permitted. Certainly the sudden occurrence of a wave over such an extensive area should be detectable experimentally despite the many other phenomena which are present in the real shear layer. One would be more persuaded of its existence if it originated in a more confined realm and then grew.

The cause of the discontinuity is u_0 departing from the real axis. It can be obviated by allowing the contour of integration to go round u_0 even when it has left the real axis. The net effect, owing to the regularity of the integrand, is to have a minute circuit round $u = u_0$ added to the integral along the real axis. A residue calculation then supplies

$$\begin{aligned}\rho_{\mathrm{r}} \sim \frac{1}{2a^2}\frac{R}{(2\pi k r_1)^{\frac{1}{2}}}\exp\left[-\mathrm{i}kr_1-\tfrac{1}{4}\pi\mathrm{i}\right] \\ +H(\gamma-\theta_1)\frac{9s^4\cos^4\gamma}{(2\pi)^{\frac{1}{2}}a^2b^2}\left\{\frac{\sin\gamma}{kr_1\sin(\gamma-\theta_1)}\right\}^{\frac{3}{2}}\exp\left[\tfrac{1}{4}\pi\mathrm{i}-\mathrm{i}kr_1\cos(\theta_1-\gamma)\right] \\ -\frac{\mathrm{i}}{a^2}\frac{H(\phi_0-\theta_1)}{\Delta'(u_0)}\{\mu_0 f(s)-\lambda_0 g(s)\}\exp\left[-\mathrm{i}ka_0\right],\end{aligned} \tag{35}$$

when $s < s_0$.

The differences between (34) and (35) are profound. Whereas the last term of (34) is present in a comparatively wide region and tends to decay exponentially, the last term of (35) is circumscribed to a relatively narrow downstream region (especially when s is near s_0) and grows exponentially. Clearly the last term of (35) is related to the Helmholtz instability of the shear layer.

What experimental evidence there is would suggest that (35) is to be preferred to (34). Admittedly, most of the experiments (Ollerhead 1966; Lowson & Ollerhead 1968; Dosanji & Yu 1969; Tam 1971; Chan & Westley 1973) are with supersonic cylindrical jets rather than the simple two dimensional subsonic shear layer considered here. But the photographs exhibit directional acoustic radiation occupying a conical region next to the flow in the downstream and Tam (1971) has verified that the angles are in accord with a cylindrical analogue of (35) for a top-hat jet. Moreover, the same directional waves can be observed in the water tank experiments described by Ffowcs Williams (1970) and which bear a closer affinity to the problem of this paper.

All things considered, therefore, it would appear that the formulae (33) and (35) should receive our backing and that (34) should be rejected. In other words, we should expect the field not to alter too dramatically with Strouhal number, with Helmholtz instability just becoming manifest in the close vicinity of the downstream part of the layer as soon as the Strouhal number is less than critical, i.e. for the longer wavelengths of incident radiation or thinner layers. Later (§ 9), analytical reasons will be adduced in support of this point of view.

The instability wave fills a sharp wedge when the Strouhal number is near critical but this wedge fans out with diminution of the Strouhal number ultimately being of angle 45° when the shear layer is very thin. This fanning out should be capable of experimental observation so that this is one test of the theory which could be instituted.

While the theory has been developed for a constant sound speed in order to reduce mathematical complication to a minimum, the evidence from previous studies dealing with instability instigated by acoustic waves suggests that a variable sound speed would not alter the phenomena qualitatively although quantitative changes are to be expected.

From another stand-point the presence of the wedge of instability wave near a jet, even when there was no incident acoustic radiation, would indicate that there were sources perturbing the flow at frequencies which made the Strouhal number less than critical. Whether the wedges due to the sources of higher frequency would be masked by those of lower frequency would depend upon the relative magnitude of the sources. Naturally, examination of the instability wave will tell us nothing about sources whose Strouhal number is above critical. In particular, no instability wave will occur for wavelengths less than six times the ratio of the width of the jet to the Mach number.

To put it another way, when attracting someone's attention in the presence of a wind, it is desirable to use lower frequencies rather than higher. Since viscosity also affects lower frequencies less than higher frequencies this is an additional argument in favour of the longer wavelength. Some other implications have already been indicated in the introduction.

Finally, there is a mathematical point to be made. It will often not be feasible to make such a complete analysis of a problem as has been achieved here, but knowledge of the instability waves may still be desired. What the above implies is that, whereas Δ has many zeros which might produce instability, it is only the one which comes off the real axis which does, in fact, generate instability. A general criterion, therefore, may well be that one need only discover those zeros which do move off the real axis as the Strouhal number varies and then assert that these provide the sole origins of instability waves notwithstanding the existence of other zeros.

In connection with this last point we may ask why the indentation round the real zero of Δ when $s > \frac{1}{2}$ cannot be drawn below by remaining on the same Riemann sheet. The field so obtained would yield a solution of the governing equations which differed from the preceding ones by a wave which was unattenuated parallel to the x-axis. The difficulty of the two zeros coming together when $s < \frac{1}{2}$ could be escaped by taking the indentation at the second zero also below. All of these solutions are mathematically acceptable in the sense that they satisfy the conditions imposed. Resolution of the non-uniqueness can be achieved either on experimental grounds or by the imposition of further conditions. We shall return to this topic in § 9 and demonstrate that by demanding that the solution be causal removes all the ambiguities and forces one to have the solution which led to (35). In the meantime we shall proceed on the basis that this is the correct solution.

4. The transmitted wave

The field produced in $y > h$, i.e. where the velocity of the main flow is again constant, is

$$\rho = -\frac{3}{2\pi a^2}\int_{-\infty}^{\infty}\frac{\xi^2 u}{\Delta}\exp\left[\mathrm{i}k(vy_0+wh-wy-ux)\right]\mathrm{d}u$$

from (24). Let $$n(u) = ux + w(y-h) - vy_0.$$

Then it may be demonstrated, as in Jones & Morgan (1972), that $n(u)$ is a univalent function of u in the upper half-plane. Thus $n'(u)$ has no zero for $\operatorname{Im} u > 0$. By taking complex conjugates we deduce that $n'(u)$ does not vanish in $\operatorname{Im} u < 0$. Hence any zeros of $n'(u)$ must lie on the real axis and it can be confirmed that there is precisely one zero, to be denoted by u'. It satisfies $-1 < u' < 1/(M+1)$ when neither $y-h$ nor y_0 is zero.

Furthermore, the curve $\operatorname{Im} n(u) = 0$ through $u = u'$ goes steadily off to infinity without crossing the real axis elsewhere. At infinity it is travelling in the direction $\arctan[\{(1-M^2)^{\frac{1}{2}}(y-h)-y_0\}/x]$ with the arctan chosen to be between 0 and π. On the curve, $\operatorname{Re} n(u)$ increases steadily as u moves away from u'. Consequently, this curve is a suitable candidate for the application of the method of stationary phase.

When the contour of integration is deformed into the curve $\operatorname{Im} n(u) = 0$ through $u = u'$, the pole $u = u_j^+$ is captured (positively) only if $\operatorname{Im} n(u_j^+) < 0$ while the pole $u = u_j^-$ is passed over (negatively) only if $\operatorname{Im} n(u_j^-) < 0$. Thus any contributions due to poles, that are crossed, are exponentially damped. In view of the discussion of the preceding section such contributions can be ignored in estimating the far field with the exception of u_0. For u_0 there are the analogues of (34) and (35) but we shall give only the version corresponding to (35).

No branch line is encountered in the deformation of the contour so that there is no field with the character of a bow wave in $y > h$ so long as $y_0 \neq 0$.

The asymptotic evaluation of the integral gives for distant points of observation

$$\rho \sim \frac{T\exp[-\mathrm{i}kn(u')+\frac{1}{4}\pi\mathrm{i}]}{2a^2(2\pi kr')^{\frac{1}{2}}} + \frac{3\mathrm{i}\,H(u',u_0)}{a^2\Delta'(u_0)}(s-khu_0)^2\exp[-\mathrm{i}kn(u_0)], \tag{36}$$

where
$$T = \frac{-6(r')^{\frac{1}{2}}(s-khu')^2u'}{\{-n''(u')\}^{\frac{1}{2}}\Delta(u')}, \quad n''(u) = \frac{y_0}{v^3} - \frac{y-h}{w^3} \tag{37}$$

and $H(u',u_0)$ signifies 1 if $\operatorname{Im} n(u) = 0$ is to the right of $u = u_0$, but zero if $\operatorname{Im} n(u) = 0$ is to the left of $u = u_0$. The quantity r' is a measure of the distance to the point of observation chosen to cancel the variation with distance of n'' so that T is independent of such variations. When $M = 0$, r' reduces to $\{x^2+(y_0+h-y)^2\}^{\frac{1}{2}}$. This artifice enables T to be regarded as a transmission coefficient.

Again, an instability wave makes its influence felt but it is now less easy to describe simply its position.

5. Plane waves

If the source of the incident sound waves is allowed to go off to infinity the radiation near the shear layer becomes a plane wave. To permit arbitrary angles of incidence change x to $x-x_0$ and put $x_0 = -r_0\cos\theta_0$, $y_0 = -r_0\sin\theta_0$ where $0 < \theta_0 < \pi$. Multiplying ρ by $2a^2(2\pi kr_0)^{\frac{1}{2}}\exp[\mathrm{i}kr_0+\frac{1}{4}\pi\mathrm{i}]$ and letting $r_0 \to \infty$ supplies the incident plane wave $\exp[-\mathrm{i}k(x\cos\theta_0+y\sin\theta_0)]$ travelling at an angle θ_0 to the positive x-axis. The term involving the reflexion coefficient R in (35) becomes $R\exp[-\mathrm{i}k(x\cos\theta_0-y\sin\theta_0)]$ with R calculated at $u = \cos\theta_0$. The transmitted wave (36) contains the term $T\exp[-\mathrm{i}kn(\cos\theta_0)]$ but now T is evaluated at $u' = \cos\theta_0$ subject to $r' = |y_0|$ being infinite.

If $\theta_0 > \gamma$, so that no bow wave can be generated, it follows from (37) that

$$1-|R|^2 = \frac{w\,|T|^2}{\sin^2\theta_0(1-M\cos\theta_0)^2}, \tag{38}$$

w having its value at $u = \cos\theta_0$. Equation (38) expresses the energy conservation law for plane waves (instability waves being disregarded) and is a generalization of one given for an abrupt discontinuity by Ribner (1957).

If $\theta_0 < \gamma$, $|R| = 1$ and total reflexion takes place. The transmitted wave now decays exponentially as y increases because $w(\cos\theta_0)$ is negative imaginary. Most of the transmitted energy is confined to a neighbourhood of $y = h$. Strictly, of course, the validity of (36) fails as θ_0 decreases to less than γ because u' is then situated in the vicinity of the branch point of w at $1/(M+1)$ and a deeper analysis is necessary to disclose the exponential attenuation of the transmitted wave when total reflexion occurs.

Naturally, these conclusions must be treated with reserve for Strouhal numbers and angles of incidence at which instability waves would be anticipated in the original problem. In spite of R and T remaining finite for these angles (the zeros of Δ/u never lie in $(-1, 1)$) the energy balance must be influenced by the energy requirements of the instability waves.

6. The thin layer

In the limit as $kh \to 0$ a discontinuous change in the flow occurs and the shear layer degenerates to a vortex sheet. It is therefore of interest to examine how the reflected and transmitted waves behave as $kh \to 0$ and compare the results with those of Jones & Morgan (1972) for the excitation of a plane vortex sheet by a line source.

It will be understood that $s \ll 1$, i.e. by a thin layer is meant that not only is the thickness of the shear layer much smaller than the acoustic wavelength but also the Mach number is not too low. Mathematically speaking, we fix M and let kh shrink to zero.

If u is bounded above then $|su| \to 0$ as $s \to 0$. In these circumstances $|\xi^2/\tau| \ll 1$ and first approximations to f and g are provided by (A 2) and (A 3). Thus

$$\Delta \approx -3\,\mathrm{i}us^2\{w + (1-Mu)^2 v\} \tag{39}$$

to a first approximation when u is bounded above. Assume that the same approximations can be introduced even if u is unrestricted. Then it is found that (34) reproduces the formula of Jones & Morgan when they do not invoke causality whereas (35) corresponds to their result when causality is imposed. There is a similar remark about the transmitted field provided that $y-h$ is replaced by y in $y > h$. In making the comparison it should be remembered that their ϕ is connected to our ρ by $\mathrm{i}\omega a^2 \rho = -\{\mathrm{i}\omega + U(y)\,\partial/\partial x\}\,\phi$.

Precise agreement with the results of Jones & Morgan can be attained, despite the possible failure of (39) for unlimited u, if it can be demonstrated that the contribution of the integral for large $|u|$ is negligible as $s \to 0$. For our purposes, this check can be circumvented by the observation that large values of u are not involved in the calculation of the distant field.

The only zero of (39) in the upper half-plane is

$$u_0 = \tfrac{1}{2}(\chi_1 + \mathrm{i}\chi_2),$$

where
$$\chi_1 = \frac{1}{M} + \left(1 + \frac{1}{M^2}\right)^{\frac{1}{2}}, \quad \chi_2 = \left(\frac{2\chi_1}{M} - 1\right)^{\frac{1}{2}}$$

to a first approximation. All the zeros u_j^+ must be of large modulus. This provides additional confirmation that u_0 is the most important of the zeros of Δ.

The inference of this section is that a thin shear layer scatters acoustic radiation in much the same way as a sharp discontinuity if $kh \ll M$.

7. The thick layer

Next consider the case where $kh \gg M$ so that s is huge. This layer may be regarded from severa different aspects. It may be thought of as a layer which is many wavelengths thick. Or the transition in velocity from 0 to U may be considered as occurring over a large distance. Or the change from 0 to U may be adjudged to be small. The last two possibilities can be covered by saying that the layer has a weak velocity gradient. Whichever point of view is adopted is immaterial provided that $kh \gg M$.

Consideration will be confined to the far field and attention will be concentrated on the determination of R and T. Therefore only values of u in the interval $(-1, 1)$ need be taken into account. Except when u is near the origin, τ will be great in magnitude when $s \gg 1$ and so asymptotic formulae will be applicable. It turns out that f and g are rather more inconvenient to work with than f_1 and g_1 (A 36, A 37) on account of (A 31). Accordingly, when $u > 0$, replace f and g by their equivalent expressions in terms of f_1 and g_1.

Begin with angles $\frac{1}{2}\pi > \theta_1 > \gamma$ so that total reflexion is absent and $u = \cos\theta_1$, satisfies $0 < u < 1/(M+1)$. Then

$$f_1(\xi) \sim \tau^{\frac{2}{3}} \exp\left[-\tfrac{1}{8}\pi\tau - \tfrac{1}{4}\mathrm{i}\tau + \tfrac{1}{4}\mathrm{i}\tau \ln \tfrac{1}{4}\tau - \tfrac{7}{8}\pi\mathrm{i} - \tfrac{2}{3}\tau z^{\frac{3}{2}}\right] \frac{\zeta}{(1-\zeta^2)^{\frac{1}{4}}} \left\{\tau - \frac{5}{48 z^{\frac{3}{2}}} - z^{\frac{1}{2}} B_0(z)\right\},$$

where $\zeta = \xi/\tau = \sec\theta_1 - M$ and z is expressed in terms of ζ via (A 16). B_0 is given by (A 34). Hence

$$u f_1'(\xi) - \mathrm{i} w f_1(\xi) \sim 2u(1-\zeta^2)^{\frac{1}{2}} f_1(\xi).$$

There is a similar result if ξ and w are replaced by s and v respectively but $u f_1'(s) + \mathrm{i} v f_1(s)$ is of lower order.

Similarly, from

$$g_1(\xi) \sim \tau^{\frac{2}{3}} \exp\left(-\tfrac{1}{8}\pi\tau + \tfrac{1}{4}\mathrm{i}\tau - \tfrac{1}{4}\mathrm{i}\tau \ln \tfrac{1}{4}\tau + \tfrac{3}{8}\pi\mathrm{i} + \tfrac{2}{3}\tau z^{\frac{3}{2}}\right) \frac{\zeta}{(1-\zeta^2)^{\frac{1}{4}}} \left\{\tau + \frac{5}{48 z^{\frac{3}{2}}} + z^{\frac{1}{2}} B_0(z)\right\},$$

$u g_1'(\xi) - \mathrm{i} w g_1(\xi)$ is of lower order.

Substitution in the formula for R gives

$$R \sim \frac{\nu_0}{2\tau(1-\zeta_0^2)^{\frac{1}{2}}} - \frac{\nu}{2\tau(1-\zeta^2)^{\frac{1}{2}}} \exp\left[\tfrac{4}{3}\tau\left(z^{\frac{3}{2}} - z_0^{\frac{3}{2}}\right)\right],$$

where $\zeta_0 = \sec\theta_1$, $\nu = (1-\frac{1}{2}\zeta^2)(1-\zeta^2)^{-\frac{5}{4}}$, ν_0 is the same with ζ_0 for ζ and z_0 is obtained from (A 16) with ζ_0 replacing ζ. In terms of the angle θ_1,

$$R \sim -\frac{\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}} \cos 2\theta_1}{4s\cos^3\theta_1 \tan^{\frac{7}{2}}\theta_1} + \frac{\cos 2\theta_1}{4s\cos\theta_1 \cos^2\theta_2 \tan^{\frac{7}{2}}\theta_2}$$
$$\times \exp\left[\tfrac{3}{4}\pi\mathrm{i} + \mathrm{i}s\{(1-M\cos\theta_1)\tan\theta_2 - \tan\theta_1 + \cos\theta_1(\operatorname{arccosh}\sec\theta_1 - \operatorname{arccosh}\sec\theta_2)\}\right], \quad (40)$$

where θ_2 is the angle between 0 and $\frac{1}{2}\pi$ such that $\sec\theta_2 = \sec\theta_1 - M$.

Correspondingly, we find

$$T \sim -\mathrm{i}\left\{\frac{r'}{-n''(u')}\right\}^{\frac{1}{2}} \frac{(1-M\cos\theta')^{\frac{1}{2}}}{\sin^{\frac{1}{2}}\theta' \sin^{\frac{1}{2}}\theta_2'}$$
$$\times \exp\left[\tfrac{1}{2}\mathrm{i}s\{(1-M\cos\theta')\tan\theta_2' - \tan\theta' + \cos\theta'(\operatorname{arccosh}\sec\theta' - \operatorname{arccosh}\sec\theta_2')\}\right], \quad (41)$$

where $$\cos\theta' = u', \ \sec\theta_2' = \sec\theta' - M$$

and now $$n''(u') = y_0 \operatorname{cosec}^3\theta' - (y-h)\cot^3\theta_2' \sec^3\theta'.$$

An extended region of validity for this formula will be indicated in the succeeding section.

It will be observed from (40) and (41) that R is an order of magnitude lower than T. Thus little energy is reflected from a thick layer at angles for which there is no bow wave. Most of the incident energy at angles for which there is no total reflexion passes through the shear layer and emerges with a transmission coefficient given by (41).

When $\theta_1 < \gamma$ or $u > 1/(1+M)$ the foregoing estimates do not hold any more since ph z alters from $-\pi$ to 0. For example, $ug_1'(\xi) - iwg_1(\xi)$ is no longer of a lower order.

When account is taken of the necessary changes we obtain

$$R \sim \exp[-\tfrac{1}{2}i\tau + \tfrac{1}{2}i\tau \ln \tfrac{1}{4}\tau - \tfrac{7}{12}\pi i + i\tau(\operatorname{arccosh} \sec\theta_1 - \sec\theta_1 \tan\theta_1)], \tag{42}$$

where $\tau = s\cos\theta_1$. Thus $|R| = 1$, as it should, in accordance with the general result of § 3 for total reflexion. It is not strictly necessary to contemplate the possibility that $u' > 1/(M+1)$ for T since it never occurs. Wherever energy would have reached the far side of the shear layer but for total reflexion the factor of T in (36) would be exponentially damped indicating that only an evanescent wave was transmitted.

Broadly speaking, therefore, we may say that the energy incident on a thick layer is either barely reflected and issues from the layer with transmission coefficient (41) or is totally reflected and, beyond the layer, is evanescent.

Some of these results should be deducible from ray theory because there are many wavelengths in a thick layer. This possibility is examined in the next section.

8. Ray theory

In ray theory the field is assumed to vary as $A_0 e^{-ik\Phi}$ where A_0 is an amplitude factor and the gradients of the phase $k\Phi$ are large compared with other gradients. Φ satisfies an eikonal equation while A_0 is determined by the conservation of energy along a ray tube. However, care is necessary when a background flow exists because the rays are, in general, not normal to the wavefronts.

Consider a ray leaving the source at the angle ψ_0 to the positive x-axis. Then, it can be shown (see, for example, Broadbent 1975) that on the ray at a point where the wave normal makes an angle ψ with the positive x-axis

$$a\sec\psi + U(y) = a\sec\psi_0. \tag{43}$$

The inclination of the ray to the x-axis at that point is $a\sin\psi\{U(y)+a\cos\psi\}^{-1}$. The differential equations governing the ray are

$$dx/dt = U(y) + a\cos\psi, \tag{44}$$

$$dy/dt = a\sin\psi. \tag{45}$$

Below $y = 0$, $U(y) \equiv 0$ and the ray is a straight line. Take $t = 0$ at the source. Then the ray strikes the interface $y = 0$ where $at = -y_0 \operatorname{cosec}\psi_0$ and $x = -y_0 \cot\psi_0$.

The change of x in the traverse of the strip $0 \leqslant y \leqslant h$ can be determined from integrating dx/dy by (44) and (45), after substituting for ψ from (43) and putting $U(y) = May/h$. The result is

$$\begin{aligned} x_1 &= \int_0^h (dx/dy)\, dy \\ &= (h/2M)\sec\psi_0[\tan\psi_0 - (1+M\cos\psi_0)\tan\psi_2 + \cos\psi_0(\operatorname{arccosh}\sec\psi_0 - \operatorname{arccosh}\sec\psi_2)], \end{aligned} \tag{46}$$

where $\sec\psi_2 = \sec\psi_0 - M$.

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Consequently, the equation of the ray in $y > h$ is

$$x = x_1 - y_0 \cot\psi_0 + (y-h)\,(M+\cos\psi_2)/\sin\psi_2. \tag{47}$$

For given ψ_0, the path of the ray is straightforward to trace but the converse of elucidating the ray which passes through a given (x, y) requires the solution of (47) for ψ_0 – a difficult matter, in general.

In a similar way, the change of t along a ray can be derived from $\mathrm{d}t/\mathrm{d}y$ integrated with respect to y. The change in at is also the variation in Φ. Initially, Φ is zero because near the source the field is

$$\frac{\exp[-\mathrm{i}kr - \frac{1}{4}\pi\mathrm{i}]}{2a^2(2\pi kr)^{\frac{1}{2}}}$$

at short wavelengths. Therefore, in $y > h$,

$$\Phi = -y_0 \operatorname{cosec}\psi_0 - (h/M)\,(\tan\psi_2 - \tan\psi_0) + (y-h)\operatorname{cosec}\psi_2. \tag{48}$$

The next hurdle to surmount is the derivation of the amplitude A_0. On a ray A_0 satisfies

$$A_0^2 l\{a + U(y)\cos\psi\}[\{U(y) + a\cos\psi\}^2 + a^2\sin^2\psi]^{\frac{1}{2}} = \text{constant},$$

where l is the perpendicular distance between two adjacent rays. To find the constant, first take the factor $\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}$ of the source to be a constant multiple to be applied to the whole. It may therefore be ignored in A_0 until the completion of the calculation. With that understanding the constant is

$$\delta(\cos\psi_0)/8\pi ka^3\sin\psi_0$$

if the inclinations of the two contiguous rays are ψ_0 and $\psi_0 + \delta\psi_0$ respectively.

The perpendicular distance between two nearby rays is given by

$$l\{(\partial f/\partial x)^2 + (\partial f/\partial y)^2\}^{\frac{1}{2}} = |\partial f/\partial\psi_0|\,\delta\psi_0$$

if the equation of a ray is $f(x, y; \psi_0) = 0$ (cf. Jones 1964). Define

$$\boldsymbol{N}(u) = n(u) - \frac{1}{2}\frac{h}{M}\left[(1-Mu)\frac{w}{u} - \frac{v}{u} + u\left\{\operatorname{arccosh}\frac{1}{u} - \operatorname{arccosh}\left(\frac{1}{u} - M\right)\right\}\right].$$

Then it may be verified in a straightforward fashion that $N'(u) = 0$ is the same as (47) when u is identified with $\cos\psi_0$. Hence, in $y > h$,

$$l\left\{1 + \left(\frac{M+\cos\psi_2}{\sin\psi_2}\right)^2\right\}^{\frac{1}{2}} = N''(\cos\psi_0)\,\delta(\cos\psi_0).$$

It follows that

$$A_0 = \frac{\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{2a^2}\left\{\frac{1 - M\cos\psi_0}{2\pi kN''(\cos\psi_0)\sin\psi_0\sin\psi_2}\right\}^{\frac{1}{2}} \tag{49}$$

the factor $\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}$ having been restored.

The field $A_0\,\mathrm{e}^{-\mathrm{i}k\Phi}$, as given by (48) and (49), is now to be compared with $T\exp[-\mathrm{i}kn(u') + \frac{1}{4}\pi\mathrm{i}]/2a^2(2\pi kr')^{\frac{1}{2}}$ with T supplied by (41). If, in $N(u)$, x is replaced by its value from (47) and u put equal to $\cos\psi_0$ then $N(u)$ is found to agree with $-\Phi$. On the other hand, $-N(u')$ is the same as the phase of the transmitted wave determined via (41). There is therefore agreement on the phases if u' is identified with $\cos\psi_0$. Recollect that (41) is a far field approximation which is equivalent to neglecting x_1 in (47) and it will be seen that the identification is legitimate. Similarly, the amplitudes are in accord to the same degree of approximation.

Thus the asymptotic analysis of § 7 and ray theory harmonize at distant points of observation. This empowers one to extend the region of validity of (41) to other points (x, y) than those originally in mind. It is only necessary to choose u' as the root of $N'(u) = 0$ instead of satisfying $n'(u) = 0$. The replacement of n'' in the denominator by N'' accomplishes the wanted modification.

There is one special case of some interest. If $\psi_0 = \frac{1}{2}\pi$, the formula for T simplifies to

$$T = -\mathrm{i}\{r'/(y-y_0)\}^{\frac{1}{2}}.$$

This expression is noteworthy for being independent of h and M. There is thus a tendency for observations in directions looking almost perpendicular to the stream to appear as if there were no stream in so far as amplitude is concerned. Amplitude measurements beyond a thick layer will be inclined to be the same whatever the flow when made in a direction transverse to the stream.

9. Causality

The question of the uniqueness of the solution to the harmonic problem has already received some attention in § 3. The matter was settled by a plausible argument coupled with an appeal to experiment. A similar difficulty arose in the discussion of the plane vortex sheet by Jones & Morgan (1972) and there the ambiguity was resolved by the physical principle of causality, namely that the answer for harmonic waves should be consistent with a field in the time domain which is zero before the acoustic excitation is switched on. In § 6 the two resolutions have been shown to be in harmony for the infinitesimally thin layer. The query to be dealt with now is whether the implications of causality would force a different solution from the one adopted if $h \neq 0$. In particular, it is desirable to know whether causality induces ultradistributions in the time domain as it does when $h = 0$.

A direct attack through a formulation in the time domain can be evaded by taking advantage of the general theory of Jones & Morgan (1974) which is based on an examination of the complex k-plane. Up to this point k has been assumed to be real and positive. Now suppose that k is permitted to have a negative imaginary part. Specifically, let $k = |k|\,\mathrm{e}^{-\mathrm{i}\alpha}$ where, for the moment, $0 \leqslant \alpha \leqslant \frac{1}{2}\pi$ so that the real part of k is not allowed to have negative values. Let us re-examine the solution already established for real k to see whether it can be adapted to cope with complex k.

Take the path of integration in (21)–(29) to be, instead of the real axis, the line obtained by rotating the axis anti-clockwise through an angle α. Thus the contour starts at $-\infty\,\mathrm{e}^{\mathrm{i}\alpha}$ and terminates at $\infty\,\mathrm{e}^{\mathrm{i}\alpha}$. There is then no difficulty in checking that the arguments justifying (21)–(29) as a solution are still of substance. A typical integral to be evaluated is

$$\rho_{\mathrm{r}} = \frac{1}{2\pi}\int_{-\infty\exp[\mathrm{i}\alpha]}^{\infty\exp[\mathrm{i}\alpha]} A(k,u)\exp[\mathrm{i}kvy - \mathrm{i}kux]\,\mathrm{d}u,$$

where the dependence of A on k and u has been explicitly indicated. The change of variable $u = \beta/k$ leads to

$$\rho_{\mathrm{r}} = \frac{1}{2\pi k}\int_{-\infty}^{\infty} A(k,\beta/k)\exp[\mathrm{i}y(k^2-\beta^2)^{\frac{1}{2}} - \mathrm{i}\beta x]\,\mathrm{d}\beta \tag{50}$$

with β real. In this form it is evident that the integral is a regular function of k subject to the proviso that the integrand has no singularities on the path of integration. The branch points of v and w do not lie on the contour unless $\alpha = 0$, a case which has already been dealt with. Therefore the only situation in which regularity can fail is when the path of integration passes through a zero of Δ. It is consequently vital to decide whether Δ has a zero on the contour when k is complex.

In effect, such a decision would depend upon a full determination of the zeros of Δ in the u-plane for every complex k – a formidable undertaking in the light of the effort required when k is real. It will, however, be sufficient for our purposes to investigate what happens as $|k| \to \infty$. In the subsequent analysis h will always be non-zero so that $|s| \to \infty$ as $|k| \to \infty$. Further τ is real and $|\tau| \to \infty$ apart from when $u \approx 0$. Thus asymptotic approximation is on the cards but the earlier analysis, based on s real and $|s/\tau|$ small, must be set aside and a fresh approach is necessary.

Consider firstly the case when τ is positive. Let $\zeta_0 = s/\tau = 1/u$ so that $0 \geqslant \mathrm{ph}\,\zeta_0 \geqslant -\frac{1}{2}\pi$. Put $\zeta = -\xi/\tau = M - 1/u$ so that ζ has a positive imaginary part. Then, if $\zeta = \cos(\sigma_r + i\sigma_i)$ as in appendix A, $\zeta \in \mathscr{D}'$ if $0 \leqslant \sigma_r \leqslant \frac{3}{4}\pi$. From (A 31), it follows that

$$\Delta \sim u^2\{f'(-\xi) + (1-\zeta^2)^{\frac{1}{2}} f(-\xi)\}[\tfrac{1}{3}\{f'(s) - (1-\zeta_0^2)^{\frac{1}{2}} f(s)\} - g'(s) + (1-\zeta_0^2)^{\frac{1}{2}} g(s)] \qquad (51)$$

for $0 \leqslant \sigma_r \leqslant \frac{3}{4}\pi$ provided that ζ is not near 0. This possibility can be excluded by imposing the condition $|M - 1/u| \geqslant \delta > 0$.

Now $$f'(-\xi) + (1-\zeta^2)^{\frac{1}{2}} f(-\xi) \sim 3\pi^{\frac{1}{2}} \tau^{\frac{5}{6}} e^{\frac{1}{4}\pi\tau} \zeta \left(\frac{1-\zeta^2}{z}\right)^{\frac{1}{4}} \{\tau^{\frac{1}{3}} z^{\frac{1}{2}} \mathrm{Ai}(\tau^{\frac{2}{3}} z) - \mathrm{Ai}'(\tau^{\frac{2}{3}} z)\}$$

where z is given by (A 16). Since u is kept away from the branch points of w, $|z|$ is bounded below and the arguments of the Airy functions are large. Asymptotic approximations can therefore be employed but the appropriate forms depend upon ph z which can range from 0 to $-\pi$. In all cases the dominant behaviour is found to be

$$f'(-\xi) + (1-\zeta^2)^{\frac{1}{2}} f(-\xi) \sim 3\tau \exp[\tfrac{1}{4}\pi\tau]\, \zeta (1-\zeta^2)^{\frac{1}{4}} \exp[-\tfrac{2}{3}\tau z^{\frac{3}{2}}] \qquad (52)$$

for $0 \geqslant \mathrm{ph}\, z \geqslant -\pi$ (compare also (A 25)). The right-hand side can vanish only if $\zeta = 0$ or $\zeta = 1$ or $\tau = 0$. The first two possibilities have already been excluded and the third one must be rejected because τ has been taken large in (52). Hence the first bracket of (51) does not vanish for the values of u permitted to date.

With regard to the last factor of (51) we remark that

$$f'(s) - (1-\zeta_0^2)^{\frac{1}{2}} f(s) = \{f'(s^*) + (1-\zeta_0^{*2})^{\frac{1}{2}} f(s^*)\}^* \qquad (53)$$

because τ is real. Proceed similarly for the g-terms. They may then be combined with the f-terms because of (A 31) since $0 \leqslant \mathrm{ph}\,\zeta_0^* \leqslant \frac{1}{2}\pi$. The result is a factor similar to the first factor of (51) in complex conjugate form. Applying (52) in the relevant guise we deduce that Δ has no zero for the values of u so far considered.

Next suppose that $-\xi/\tau$ gives a σ_r which exceeds $\frac{3}{4}\pi$ so that it may not be in $\mathscr{D}'$. Here it is advantageous to operate with ξ/τ and to put $\zeta = -M + 1/u$, so that now $0 \geqslant \mathrm{ph}\,\zeta \geqslant -\frac{1}{2}\pi$. In the same way as (53) was derived

$$(\Delta/u^2)^* = \{g'(\xi^*) - (1-\zeta^{*2})^{\frac{1}{2}} g(\xi^*)\}\{f'(s^*) + (1-\zeta_0^{*2})^{\frac{1}{2}} f(s^*)\}$$
$$-\{f'(\xi^*) - (1-\zeta^{*2})^{\frac{1}{2}} f(\xi^*)\}\{g'(s^*) + (1-\zeta_0^{*2})^{\frac{1}{2}} g(s^*)\}$$

and $0 \leqslant \mathrm{ph}\,\zeta^* \leqslant \frac{1}{2}\pi$. Let $\zeta^* = \cos(\sigma_r + i\sigma_i)$, $\zeta_0^* = \cos(\sigma_{r0} + i\sigma_{i0})$ so that $0 \leqslant \sigma_r \leqslant \frac{1}{2}\pi$ though $0 \leqslant \sigma_r \leqslant \frac{1}{4}\pi$ will suffice for our needs. Then $\sigma_r > \sigma_{r0}$ and, since $\mathrm{Im}\,\zeta^* = \mathrm{Im}\,\zeta_0^*$, $\sigma_i > \sigma_{i0}$. If

$$\tfrac{2}{3} z^{\frac{3}{2}} = \tfrac{1}{2}(\sigma_r + i\sigma_i) - \tfrac{1}{4}\sin 2(\sigma_r + i\sigma_i)$$

and $$\tfrac{2}{3} z_0^{\frac{3}{2}} = \tfrac{1}{2}(\sigma_{r0} + i\sigma_{i0}) - \tfrac{1}{4}\sin 2(\sigma_{r0} + i\sigma_{i0}),$$

$$\mathrm{Re}\,\tfrac{4}{3}(z^{\frac{3}{2}} - z_0^{\frac{3}{2}}) = \sigma_r - \sigma_{r0} - \tfrac{1}{2}\sin 2\sigma_r \cosh 2\sigma_i + \tfrac{1}{2}\sin 2\sigma_{r0} \cosh 2\sigma_{i0}.$$

But $$\sin 2\sigma_r \sinh^2 \sigma_i = \cot \sigma_r \tan \sigma_{r0} \sin 2\sigma_{r0} \sinh^2 \sigma_{i0}$$

since $\operatorname{Im} \zeta^* = \operatorname{Im} \zeta_0^*$. Further $0 \leqslant \tan \sigma_{r0} \cot \sigma_r < 1$ for $0 \leqslant \sigma_{r0} < \sigma_r \leqslant \frac{1}{2}\pi$. Hence

$$\operatorname{Re} \tfrac{4}{3}(z^{\frac{3}{2}} - z_0^{\frac{3}{2}}) > \sigma_r - \tfrac{1}{2}\sin 2\sigma_r - \sigma_{r0} + \tfrac{1}{2}\sin 2\sigma_{r0} > 0$$

since $x - \sin x$ is an increasing function of x.

Evidently, terms with exponent $\frac{2}{3}\tau(z^{\frac{3}{2}} - z_0^{\frac{3}{2}})$ in $(\Delta/u^2)^*$ will dominate those with exponent $-\frac{2}{3}\tau(z^{\frac{3}{2}} - z_0^{\frac{3}{2}})$. Before substituting uniformly valid asymptotic formulae it is convenient to replace g by g_1 via (A 37). The structure of $(\Delta/u^2)^*$ is unaltered apart from multiplication by a non-zero constant. Taking account of possible variations of ph z we find that in all cases $(\Delta/u^2)^*$ is a non-zero multiple of $\exp \frac{2}{3}\tau(z^{\frac{3}{2}} - z_0^{\frac{3}{2}})$. Again, Δ has no zero.

It has thus been established that Δ has no zero for positive τ subject to the restrictions $|u| > \delta$, $|u| < 1/\delta$ and $|u - 1/M| > \delta$.

For negative values of τ, first replace τ in the second arguments of f and g by $|\tau|$, a process which has no effect on f and g. Choose now $\zeta_0 = s/|\tau|$ and $\zeta = M + \mathrm{e}^{-\mathrm{i}\alpha}/|u|$. Then

$$uf'(s) + \mathrm{i}vf(s) = u\{f'(s^*) - (1 - \zeta_0^{*2})^{\frac{1}{2}} f(s^*)\}^*$$

and there is a similar form for the terms involving ξ. Consequently we have the same complex conjugate form of Δ as that just discussed but with s^* and ξ^* interchanged. Moreover, $\operatorname{Re}(z^{\frac{3}{2}} - z_0^{\frac{3}{2}}) < 0$. On this occasion, we obtain a non-zero multiple of $\exp \frac{2}{3}|\tau|(z_0^{\frac{3}{2}} - z^{\frac{3}{2}})$ as the dominant term. Once again Δ has no zero.

It has now been demonstrated that the only places where Δ might have a zero on the contour of integration are contained in the regions $|u| < \delta$, $|u| > 1/\delta$ and $|u - 1/M| < \delta$. The first of these occurs because the preceding analysis demanded that $|\tau| \gg 1$. It can be made arbitrarily small by prescribing it as $|u| < |s|^{-\frac{1}{2}}$ which would continue to keep $|\tau|$ large outside. But near the origin (A 43)–(A 46) imply that the only zero actually coincides with the origin. However, such a zero is automatically cancelled in the integrals by a corresponding one in the numerator. The region $|u| < \delta$ may therefore be discarded.

For $|u| > 1/\delta$, the appropriate formula is (51) but now the asymptotic expression for $f(s)$ goes awry because ζ_0 can approach the singularity at the origin. This difficulty can be overcome by using (A 58)–(A 60) which are uniformly valid as $\kappa = s/\tau \to 0$. The result is that the second factor of (51) is a non-zero multiple of $(4\tau Z^{\frac{1}{2}} - 1)$ where Z is given by (A 57). This can disappear only if $Z^{\frac{1}{2}}$ is very small but then $Z^{\frac{1}{2}} \approx \frac{1}{2}\kappa$ and the expression reduces to $2s - 1$ which cannot be zero because $|s| \gg 1$. Hence Δ has no zeros in $|u| > 1/\delta$ if τ is positive. For negative τ the application of the complex conjugate technique already employed directs one to the same conclusion.

The only outstanding region where there may be a zero is $|u - 1/M| < \delta$. Handling this in the ame way as $|u| > 1/\delta$ we discover that Δ is a non-zero multiple of

$$4\tau Z^{\frac{1}{2}} \exp(2\tau Z^{\frac{1}{2}}) - \exp(-2\tau Z^{\frac{1}{2}}),$$

where now $\kappa = M - 1/u$. This expression vanishes when $4\tau Z^{\frac{1}{2}} = t_0$ where t_0 is the positive zero of $t\,\mathrm{e}^t = 1$. Consequently, $Z^{\frac{1}{2}} \approx \frac{1}{2}\kappa$ and the zero lies where $Mu = 1 + t_0/2s$. For real s, this gives a real zero of Δ near $1/M$ in conformity with what is known from appendix B. Only small values of α occur in the region. When $\alpha \neq 0$, the zero is above the real u-axis but below the path of integration which is along the radius vector from the origin through $\mathrm{e}^{\mathrm{i}\alpha}$. We deduce that this zero causes no trouble so long as it is skirted by an indentation above when s is real.

Accordingly, it has been proved that Δ has no zero of relevance on the contour of integration if $|k|$ is sufficiently large, say $|k| > K$, and $0 \leqslant \mathrm{ph}\, k \leqslant \frac{1}{2}\pi$. This establishes that the integrals for ρ are regular functions of k in this part of the k-plane. It remains to tackle the topic of k having a negative real part.

Let k be purely imaginary and $\mathrm{Im}\, k < -K$. Then (50) holds with $k = -\mathrm{i}|k|$ and provides a continuous function of k. Furthermore, a change in the sign of β gives

$$\rho_{\mathrm{r}} = \frac{-1}{2\pi \mathrm{i}|k|}\int_{-\infty}^{\infty} A(-\mathrm{i}|k|,\, -\mathrm{i}\beta/|k|)\exp\{y(|k|^2+\beta^2)^{\frac{1}{2}}+\mathrm{i}\beta x\}\,\mathrm{d}\beta.$$

This is the complex conjugate of (50) if

$$\{A(-\mathrm{i}\,|k|, \mathrm{i}\beta/|k|)\}^* = -A(-\mathrm{i}\,|k|,\, -\mathrm{i}\beta/|k|)$$

or, equivalently,
$$\{A(-\mathrm{i}\,|k|, u)\}^* = -A(-\mathrm{i}\,|k|, u^*) \tag{54}$$
when u is purely imaginary.

In these circumstances
$$\{f(s,\tau)\}^* = f(s^*,\tau) = -f(s,\, -\tau)$$

remembering that τ is real. Moreover

$$\{f(\xi,\tau)\}^* = f(s^*(1-Mu^*),\tau) = -f(s(1-Mu^*),\, -\tau)$$

and $\{v(u)\}^* = v(u^*)$, $\{w(u)\}^* = w(u^*)$. By using formulae of this type and observing that v is real we can confirm that (54) holds. It follows that ρ_{r} is real on the negative imaginary k-axis when $\mathrm{Im}\, k < -K$.

Likewise, it may be checked that (24) and (27) supply real values of ρ on the negative imaginary k-axis as well.

The integrals discussed so far are all functions of k. Denote a typical one by $\rho(k)$. Define $\rho(k)$ for $\mathrm{Re}\, k < 0$ by
$$\rho(k) = \{\rho(-k^*)\}^*. \tag{55}$$

The field so defined is continuous and real on $\mathrm{Re}\, k = 0$, $\mathrm{Im}\, k < -K$. Therefore, by the Riemann–Schwarz principle of reflexion (Titchmarsh 1939), (55) provides an analytic continuation into $\mathrm{Re}\, k < 0$. Thus ρ, as defined by (55), is regular throughout $\mathrm{Im}\, k < -K$. Indeed, since regularity has been demonstrated for $\mathrm{Re}\, k > 0$, $|k| > K$ the definition (55) supplies a regular function of k in $|k| > K$, $\mathrm{Im}\, k \leqslant 0$. Its only singularities in $\mathrm{Im}\, k \leqslant 0$ will be at certain points in $|k| < K$, $\mathrm{Re}\, k > 0$ and their mirror images in the imaginary k-axis.

It is evident that the integrals tend to zero as $\mathrm{Im}\, k \to -\infty$. Hence ρ, as defined, is a regular function of k in $\mathrm{Im}\, k < -K$ which tends to zero as $\mathrm{Im}\, k \to -\infty$. It is therefore certain that ρ is the Fourier transform of a causal solution in the time domain. Allow k to become real and positive while staying the region $|k| > K$. Then, no singularities of ρ are encountered and the integral changes steadily to its value with $\alpha = 0$. This means that the integrals accepted in § 3, for example, are recovered if $k > K$. In other words the integrals of § 3 certainly originate from a causal solution in the time domain if $k > K$. From our knowledge of the behaviour for real k we can be sure of their correctness for $s > s_0$.

The statements just made depend upon it being right to place the indentation at the real positive zero of Δ in the u-plane above the real axis. This procedure has already been justified by our discussion of the region $|u - 1/M| < \delta$ which shows that, as $\mathrm{Im}\, k \to 0$, the zero tends to the real u-axis from below the contour of integration. Had the approach to the real u-axis been above the path of integration an additional wave would have made its appearance in $s > s_0$.

It is not possible to deduce immediately the solution for $s < s_0$ from that for $s > s_0$ by keeping k real because the fusing of the two real zeros in the u-plane is responsible for the creation of a branch point in the k-plane roughly of the form $(k - Ms_0/h)^{-\frac{1}{2}}$. The correct choice of branch is indeterminate if real values alone of k are considered.

The points where $s < s_0$ or $k < Ms_0/h$ cannot be reached from $\mathrm{Im}\, k < -K$ without crossing the region $|k| < K$ where the integrals may possess singularities, due to poles appearing on the contour of integration. However, the function will vary continuously unless no route can be found in the k-plane which arrives at the real axis without intersecting a branch line of the function. The singularities of the function occur in pairs which are mirror images in the imaginary k-axis. Therefore, if one of the pair is a branch point so is the other. Hence the branch line from one terminates at the other since it certainly cannot intrude on $|k| > K$. The branch line may be conveniently drawn as the straight line joining the two points. Thus, by selecting a track which circles any branch points or isolated singularities we can alight at the real axis with continuous variations in the function and the real axis must be approached from below.

On this basis, the solution for $s < s_0$ must be obtained from that for $s > s_0$ by continuous variation as s by-passes s_0 by a trajectory in the lower half of the complex s-plane. This implies that $u = u_0$ must be surrounded by a small circuit as it leaves the real u-axis and that (35) is the correct formula for $s < s_0$ according to causal considerations.

The contrast between the cases $h = 0$ and $h \neq 0$ is strong. If $h = 0$ and the layer is abrupt, causality cannot be satisfied without the introduction of ultradistributions in the time domain. Correspondingly, Helmholtz instability is manifest in the frequency domain and exponentially growing waves are present. There are also instability waves in the frequency domain if $h \neq 0$ but they are not related to ultra-distributions in the time domain because they disappear when s is big enough. Accordingly, they correspond to conventional functions and there is no awkwardness in physical interpretation.

At first sight it may perhaps be unexpected that, although the instability wave in the frequency domain for $h \neq 0$ tends to that for $h = 0$ as $h \to 0$, ultradistributions are not necessary in the time domain for both cases. To illustrate how this can happen we consider the simple example in which the instability wave has the special form $\exp[-\mathrm{i}kdx]$ where d is a complex constant and k is positive. For negative k the rule (55) makes it $\exp[-\mathrm{i}kd^*x]$. Assuming that it is present only for $|k| < C$ when $h \neq 0$, we obtain for the corresponding field in the time domain

$$\int_0^C \exp[\mathrm{i}k(at - dx)]\,\mathrm{d}k + \int_{-C}^0 \exp[\mathrm{i}k(at - d^*x)]\,\mathrm{d}k$$
$$= \frac{2}{b_1}\exp[\tfrac{1}{2}\mathrm{i}Cb_1]\sin\tfrac{1}{2}Cb_1 + \frac{2}{b_1^*}\exp[-\tfrac{1}{2}\mathrm{i}Cb_1^*]\sin\tfrac{1}{2}Cb_1^*, \tag{56}$$

where $b_1 = at - dx$. If $C \to \infty$ as $h \to \infty$, the instability wave becomes $\exp[-\mathrm{i}kdx]$ for all positive k in the frequency domain but the right-hand side of (56) becomes the ultradistribution

$$\pi\{\delta(b_1) + \delta(b_1^*)\} + \mathrm{i}\{b_1^{-1} - (b_1^*)^{-1}\}.$$

This illustrates the different effect of the limiting operation in the two domains.

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10. Conclusion

Our discussion of the simple shear layer has exposed the fact that there is a critical Strouhal number below which Helmholtz instability occurs. The instability corresponds to a conventional function in the time domain and experimental tests for the model have been conjectured in § 3. In particular, the continuous variation of the field with complex Strouhal number is at the heart of the matter.

It has been shown that as the thickness of the layer shrinks the solution goes over to that for the vortex sheet. The vortex sheet may therefore be regarded as an adequate model for the very thin layer.

Complete agreement between ray theory and the exact theory in the regions where rays penetrate has been accomplished for layers with weak velocity gradients.

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Appendix A

This appendix is devoted to a discussion of the properties of solutions of

$$\frac{d^2\tilde{p}}{d\xi^2}-\frac{2}{\xi}\frac{d\tilde{p}}{d\xi}-\left(1-\frac{\xi^2}{\tau^2}\right)\tilde{p}=0 \tag{A 1}$$

in the complex ξ- and τ-planes. For fixed τ, (A 1) has a regular singularity at $\xi = 0$, the exponents being 0 and 3. The usual series expansion gives two independent solutions which will be denoted by f and g. The leading terms of the series expansions, which are convergent in the whole finite part of the ξ-plane, are

$$f(\xi,\tau) = \xi^3+\tfrac{1}{10}\xi^5+\frac{1}{28}\left(\frac{1}{10}-\frac{1}{\tau^2}\right)\xi^7+\ldots, \tag{A 2}$$

$$g(\xi,\tau) = 1-\tfrac{1}{2}\xi^2-\frac{1}{4}\left(\frac{1}{2}+\frac{1}{\tau^2}\right)\xi^4+\ldots. \tag{A 3}$$

The expansions converge for all finite complex τ, apart from $\tau = 0$ which is an essential singularity. With this exception it is clear that

$$f(-\xi,\tau) = -f(\xi,\tau), \quad g(-\xi,\tau) = g(\xi,\tau), \tag{A 4}$$

$$f(\xi,-\tau) = f(\xi,\tau), \quad g(\xi,-\tau) = g(\xi,\tau). \tag{A 5}$$

When no confusion will arise, the explicit dependence of f and g on τ will be dropped and the notation $f(\xi)$ and $g(\xi)$ adopted.

The solutions can also be expressed in terms of Whittaker functions or of confluent byper-geometric functions. The relevant formulae are

$$f(\xi) = \exp[-\tfrac{5}{8}\pi\mathrm{i}]\,\tau^{\frac{5}{4}}\xi^{\frac{1}{2}}M_{\frac{1}{4}\mathrm{i}\tau,\frac{3}{4}}\left(\frac{\mathrm{i}\xi^2}{\tau}\right) = \xi^3\Phi\left(\tfrac{5}{4}-\tfrac{1}{4}\mathrm{i}\tau,\tfrac{5}{2};\frac{\mathrm{i}\xi^2}{\tau}\right)\exp\left(-\frac{\mathrm{i}\xi^2}{2\tau}\right), \tag{A 6}$$

$$g(\xi) = \exp[\tfrac{1}{8}\pi\mathrm{i}]\,\tau^{-\frac{1}{4}}\xi^{\frac{1}{2}}M_{\frac{1}{4}\mathrm{i}\tau,-\frac{3}{4}}\left(\frac{\mathrm{i}\xi^2}{\tau}\right)$$

$$= \Phi\left(-\tfrac{1}{4}-\tfrac{1}{4}\mathrm{i}\tau,\ -\tfrac{1}{2};\frac{\mathrm{i}\xi^2}{\tau}\right)\exp\left(-\frac{\mathrm{i}\xi^2}{2\tau}\right) \tag{A 7}$$

in standard notation (see, for example, Erdélyi 1953).

It is an easy deduction from (A 1), (A 2) and (A 3) that the Wronskian

$$f(\xi)\,g'(\xi) - f'(\xi)\,g(\xi) = -3\xi^2 \tag{A 8}$$

primes indicating derivatives with respect to ξ.

Also $$\{f(\xi)f'(\xi)/\xi^2\}' = f'^2/\xi^2 + (1-\xi^2/\tau^2)f^2/\xi^2$$

so that, if ξ and τ are real with $\xi^2 \leqslant \tau^2$, ff'/ξ^2 is an increasing function of ξ. Since it is zero at $\xi = 0$, it must be positive for $0 < \xi$ i.e. f and f' are both positive for $0 < \xi \leqslant \tau$. Furthermore, since

$$(\xi f' - 3f)' = \xi(1-\xi^2/\tau^2)f,$$

we conclude that $$f' \geqslant 3f/\xi, \quad f \geqslant \xi^3, \quad f' \geqslant 3\xi^2 \tag{A 9}$$

on $0 \leqslant \xi \leqslant \tau$.

Similarly, gg'/ξ^2 is an increasing function of ξ. It starts from negative infinity at $\xi = 0$ and so can change sign at most once on $0 < \xi \leqslant \tau$. Therefore, at most one of g and g' can pass through a zero as ξ goes positively up to τ from the origin.

The behaviour near the essential singularity at $\tau = 0$ can be derived from the asymptotic formula for the confluent bypergeometric function. As $|z| \to \infty$

$$\Phi(a,c;z) = \frac{(c-1)!}{(a-1)!}\mathrm{e}^z z^{a-c}\left\{1+(c-a)(1-a)\frac{1}{z}+O\left(\frac{1}{|z|^2}\right)\right\}$$

$$+\frac{(c-1)!}{(c-a-1)!}\left(\frac{\mathrm{e}^{\mathrm{i}\pi}}{z}\right)^a\left\{1-a(a-c+1)\frac{1}{z}+O\left(\frac{1}{|z|^2}\right)\right\} \tag{A 10}$$

for $0 \leqslant \mathrm{ph}\,z < \pi$ where ph denotes the phase or argument of z. Hence, if ξ is fixed and $\mathrm{ph}\,(\tau/\xi^2) = 0$, (A 6) and (A 7) supply

$$f(\xi) \approx (\tfrac{3}{2}!/\tfrac{1}{4}!)\,2\xi^{\frac{1}{2}}\tau^{\frac{5}{4}}\cos(\tfrac{5}{8}\pi - \tfrac{1}{2}(\xi^2/\tau)), \tag{A 11}$$

$$f'(\xi) \approx (\tfrac{3}{2}!/\tfrac{1}{4}!)\,2\xi^{\frac{3}{2}}\tau^{\frac{1}{4}}\sin(\tfrac{5}{8}\pi - \tfrac{1}{2}(\xi^2/\tau)), \tag{A 12}$$

$$g(\xi) \approx \{(-\tfrac{3}{2})!/(-\tfrac{5}{4})!\}\,2\xi^{\frac{1}{2}}\tau^{-\frac{1}{4}}\cos(\tfrac{1}{8}\pi + \tfrac{1}{2}(\xi^2/\tau)), \tag{A 13}$$

$$g'(\xi) \approx -\{(-\tfrac{3}{2})!/(-\tfrac{5}{4})!\}\,2\xi^{\frac{3}{2}}\tau^{-\frac{5}{4}}\sin(\tfrac{1}{8}\pi + \tfrac{1}{2}(\xi^2/\tau)) \tag{A 14}$$

as $\tau \to 0$. These approximations are not reliable when ph $(\tau/\xi^2) \neq 0$ because the neglected terms can then be exponentially large and of the same order as the terms retained. Some formulae which do not have this failing will be given later (cf. (A 43)).

The performance as $|\tau| \to \infty$ is less simple because it is necessary to allow in our applications for ξ being large and complex as well. Some results concerning the asymptotic behaviour of Whittaker functions can be found in Erdélyi & Swanson (1957), Olver (1974), but they do not cover the range of parameters required here. The appropriate formulae can, however, be determined by means of the technique due to Olver (1954). No attempt is made here to calculate error bounds although they could be obtained by the method described by Olver (1974).

Make the change of variable $\xi = \zeta\tau$ and put $f = \zeta w$. Then the differential equation for f transforms to

$$\mathrm{d}^2w/\mathrm{d}\zeta^2 + \left\{\tau^2(\zeta^2-1) - \frac{2}{\zeta^2}\right\} w = 0. \qquad \text{(A 15)}$$

Introduce the variable z, defined by

$$z^{\frac{1}{2}}\,\mathrm{d}z/\mathrm{d}\zeta = -(1-\zeta^2)^{\frac{1}{2}}$$

so that

$$\tfrac{2}{3}z^{\frac{3}{2}} = -\int_1^{\zeta}(1-t^2)^{\frac{1}{2}}\,\mathrm{d}t = \tfrac{1}{2}\arccos\zeta - \tfrac{1}{2}\zeta(1-\zeta^2)^{\frac{1}{2}}. \qquad \text{(A 16)}$$

Apply the Liouville transformation

$$w = (\mathrm{d}z/\mathrm{d}\zeta)^{-\frac{1}{2}}\,W$$

to (A 15) and then

$$\mathrm{d}^2W/\mathrm{d}z^2 = \{\tau^2 z + h(z)\}\,W, \qquad \text{(A 17)}$$

where

$$h(z) = \frac{2}{\zeta^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^2 + \left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^{\frac{1}{2}}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left(\frac{\mathrm{d}\zeta}{\mathrm{d}z}\right)^{-\frac{1}{2}}$$

$$= \frac{2z}{\zeta^2(1-\zeta^2)} + \frac{5}{16z^2} - \frac{z(2+3\zeta^2)}{4(1-\zeta^2)^3}. \qquad \text{(A 18)}$$

The mapping (A 16) is multiple-valued. To achieve a precise relationship let $\sigma = \sigma_r + \mathrm{i}\sigma_i$, where σ_r and σ_i are real. Put

$$\zeta = \cos\sigma, \quad (1-\zeta^2)^{\frac{1}{2}} = \sin\sigma,$$

$$\tfrac{2}{3}z^{\frac{3}{2}} = \tfrac{1}{2}\sigma - \tfrac{1}{4}\sin 2\sigma.$$

The upper half of the ζ-plane corresponds to $0 < \sigma_r < \pi$, $\sigma_i \leqslant 0$ and the lower half to $-\pi < \sigma_r < 0$, $\sigma_i \leqslant 0$. In fact, we shall not be concerned with the whole of the ζ-plane. Instead, a cut will be inserted in the z-plane along the positive real axis from $(3\pi)^{\frac{2}{3}}/4$ to ∞. The range of ph z will then be specified by $-2\pi <$ ph $z < 0$. The region of the z-plane below the real axis then maps one-to-one on to the upper half of the ζ-plane limited on the left by the image of the cut. This image runs along the negative real axis from $\zeta = 0$ to $\zeta = -1$ and then follows a curve in the upper half-plane until it is eventually travelling towards $\infty\,\mathrm{e}^{\frac{3}{4}\pi\mathrm{i}}$. Its exact location off the real axis is not relevant to our investigation but it is certainly to the left of the hyperbola $\sigma_r = \frac{3}{4}\pi$. Similarly, the remaining region of the z-plane is mapped into the lower half of the ζ-plane bounded on the left by the image of the cut. In this case, the curved portion is eventually moving in the direction of $\infty\,\mathrm{e}^{-\frac{3}{4}\pi\mathrm{i}}$ and is to the left of $\sigma_r = -\frac{3}{4}\pi$. Denote by $\mathscr{D}$ the region of the ζ-plane which is mapped one-to-one onto the z-plane by the above process.

It may be verified readily from (A 18) that $h(z)$ is regular in the cut z-plane and $O(|z|^{-2})$, uniformly with respect to ph z, as $|z| \to \infty$. The preliminary conditions of Olver's theory are therefore met when τ is real and positive. Hence, it may be asserted that there is a solution W_1 of (A 17), such that, as $\tau \to \infty$,

$$W_1(z) = \mathrm{Ai}(\tau^{\frac{2}{3}}z)\left\{1+\sum_{s=1}\frac{A_s(z)}{\tau^{2s}}\right\}+\frac{\mathrm{Ai}'\,(\tau^{\frac{2}{3}}z)}{\tau^{\frac{4}{3}}}\sum_{s=0}\frac{B_s(z)}{\tau^{2s}} \tag{A 19}$$

where Ai (x) is the standard Airy function and

$$B_s(z) = \frac{1}{2z^{\frac{1}{2}}}\int_0^z t^{-\frac{1}{2}}\{h(t)\,A_s(t)-A_s''(t)\}\,\mathrm{d}t, \tag{A 20}$$

$$A_{s+1}(z) = -\tfrac{1}{2}B_s'(z)+\tfrac{1}{2}\int h(t)\,B_s(t)\,\mathrm{d}t$$

subject to the starting condition $A_0(z) = 1$.

For (A 19) to be valid the area above the cut and below the level curve of $\exp\,(-\frac{2}{3}z^{\frac{3}{2}})$ through the end of the cut must be excluded. In effect, this exclusion removes the remainder of the third quadrant from $\mathscr{D}$. Let $\mathscr{D}'$ be the domain so obtained.

Now investigate the possibility that, as $\tau \to \infty$,

$$f(\zeta\tau,\tau) = A\zeta(z/(1-\zeta^2))^{\frac{1}{4}}W_1(z),$$

A being independent of ζ which lies in $\mathscr{D}'$. The constant A is identified by allowing $\zeta \to \infty$ along the positive real axis. From (A 6) and (A 10),

$$f(\zeta\tau,\tau) \sim 3\zeta^{\frac{1}{2}}\tau\,\mathrm{e}^{\frac{1}{4}\pi\tau}\cos\,(\tfrac{1}{2}\zeta^2\tau-\tfrac{1}{4}\pi-\tfrac{1}{4}\tau-\tfrac{1}{2}\tau\ln 2\zeta) \tag{A 21}$$

after employing Stirling's asymptotic formula for the factorial functions involving τ.On the other hand, as $\zeta \to \infty$,

$$\tfrac{2}{3}z^{\frac{3}{2}} \sim (\tfrac{1}{2}\zeta^2-\tfrac{1}{2}\ln 2\zeta-\tfrac{1}{4})\,\mathrm{e}^{-\frac{3}{2}\pi\mathrm{i}}$$

and hence

$$A\zeta(z/(1-\zeta^2))^{\frac{1}{4}}W_1(z) \sim (A\zeta^{\frac{1}{2}}/\pi^{\frac{1}{2}}\tau^{\frac{1}{6}})\cos\,(\tfrac{1}{2}\zeta^2\tau-\tfrac{1}{4}\pi-\tfrac{1}{4}\tau-\tfrac{1}{2}\tau\ln 2\zeta). \tag{A 22}$$

It follows from (A 21) and (A 22) that $A = 3\pi^{\frac{1}{2}}\tau^{\frac{7}{6}}\,\mathrm{e}^{\frac{1}{4}\pi\tau}$ and therefore

$$f(\zeta\tau,\tau) \sim 3\pi^{\frac{1}{2}}\tau^{\frac{7}{6}}\,\mathrm{e}^{\frac{1}{4}\pi\tau}\zeta(z/(1-\zeta^2))^{\frac{1}{4}}\,\mathrm{Ai}\,(\tau^{\frac{2}{3}}z) \tag{A 23}$$

as $\tau \to \infty$ with ζ in $\mathscr{D}'$. A higher order approximation is achieved by replacing Ai in (A 23) by W_1, as given by (A 19), and calculating the coefficients A_s and B_s from (A 20).

The general theory implies that

$$f'(\zeta\tau,\tau) \sim -3\pi^{\frac{1}{2}}\tau^{\frac{5}{6}}\,\mathrm{e}^{\frac{1}{4}\pi\tau}\zeta\left(\frac{1-\zeta^2}{z}\right)^{\frac{1}{4}}\mathrm{Ai}'\,(\tau^{\frac{2}{3}}z) \tag{A 24}$$

as $\tau \to \infty$ with ζ in $\mathscr{D}'$. It is worth noting that, because of the asymptotic behaviour of the Airy function and its derivative,

$$f'(\zeta\tau,\tau) \sim (1-\zeta^2)^{\frac{1}{2}}f(\zeta\tau,\tau) \tag{A 25}$$

whenever $|z| \geqslant \delta > 0$, $0 \geqslant \mathrm{ph}\,z \geqslant -\frac{2}{3}\pi$ and $\mathrm{Im}\,\zeta \geqslant 0$. Also

$$f'(\zeta\tau,\tau) \sim -(1-\zeta^2)^{\frac{1}{2}}f(\zeta\tau,\tau) \tag{A 26}$$

for $|z| \geqslant \delta > 0$, $-\frac{4}{3}\pi \geqslant \mathrm{ph}\,z \geqslant -2\pi$ and $\mathrm{Im}\,\zeta \leqslant 0$.

Both (A 23) and (A 24) hold uniformly with respect to ζ in $\mathscr{D}'$. Asymptotic formulae for a somewhat larger region can be obtained immediately, when τ is real, from the observation that $f(\xi^*) = \{f(\xi)\}^*$ for real τ, the star signifying a complex conjugate.

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When τ is complex let it be $\tau_0\,\mathrm{e}^{\mathrm{i}\theta}$ where τ_0 is positive; for the moment, attention will be devoted to the case where $|\theta| \leqslant \frac{1}{2}\pi$. Putting $z = z_0\,\mathrm{e}^{-\frac{2}{3}\mathrm{i}\theta}$ converts (A 17) to

$$\mathrm{d}^2W/\mathrm{d}z_0^2 = \{\tau_0^2 z_0 + \mathrm{e}^{-\frac{4}{3}\mathrm{i}\theta}\, h(z_0\,\mathrm{e}^{-\frac{2}{3}\mathrm{i}\theta})\}\, W.$$

Because of the similarity in form to (A 17), the same treatment may be applied. The cut in the z_0-plane is that of the z-plane but rotated anti-clockwise through an angle $2\theta/3$. In the ζ-plane we continue to work with the domain $\mathscr{D}'$. There is still an expansion of the form (A 19), but in terms of τ_0 and z_0. We try to discover the appropriate form for f by making $z_0 \to \infty\,\mathrm{e}^{-\pi\mathrm{i}}$. The formula (A 21) is valid for $\theta > 0$ provided that θ is not near $\frac{1}{2}\pi$.

In this way, we arrive at the important conclusion that (A 23) and (A 24) are valid uniformly for ζ in $\mathscr{D}'$ as $|\tau| \to \infty$ with $0 \leqslant \mathrm{ph}\,\tau \leqslant \frac{1}{2}\pi - \delta$, $\delta > 0$.

When θ is near $\frac{1}{2}\pi$, (A 21) fails because of the change in the asymptotic behaviour of the factorial function. For θ near $\frac{1}{2}\pi$, (A 21) must be replaced by

$$f(\zeta\tau,\tau) \sim 3\zeta^{\frac{1}{2}}\tau\,\mathrm{e}^{\frac{1}{4}\pi\tau}(1-\mathrm{i}\,\mathrm{e}^{-\frac{1}{2}\pi\tau})\cos(\tfrac{1}{2}\zeta^2\tau - \tfrac{1}{4}\pi - \tfrac{1}{4}\tau - \tfrac{1}{2}\tau\ln 2\zeta)$$
$$+\tfrac{3}{2}\zeta^{\frac{1}{2}}\tau\exp(\tfrac{1}{4}\pi\mathrm{i} - \tfrac{1}{4}\pi\tau - \tfrac{1}{4}\mathrm{i}\tau + \tfrac{1}{2}\mathrm{i}\zeta^2\tau - \tfrac{1}{2}\mathrm{i}\tau\ln 2\zeta).$$

A representation in terms of $\mathrm{Ai}\,(\tau_0^{\frac{2}{3}} z_0)$ alone is no longer adequate and a term involving $\mathrm{Ai}(\tau_0^{\frac{2}{3}} z_0\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})$ must be added. Observing that, as $z_0 \to \infty\,\mathrm{e}^{-\mathrm{i}}$, $\zeta \to \infty\,\mathrm{e}^{-\frac{1}{2}\mathrm{i}\theta}$ and so is well within $\mathscr{D}'$ we have

$$f(\zeta\tau,\tau) \sim 3\pi^{\frac{1}{2}}\tau^{\frac{7}{6}}\zeta\left(\frac{z}{1-\zeta^2}\right)^{\frac{1}{4}}\{\mathrm{e}^{\frac{1}{4}\pi\tau}\,\mathrm{Ai}(\tau^{\frac{2}{3}}z) + \mathrm{e}^{-\frac{1}{6}\pi\mathrm{i}-\frac{1}{4}\pi\tau}\,\mathrm{Ai}(\tau^{\frac{2}{3}}z\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})\}, \qquad \text{(A 27)}$$

uniformly for ζ in $\mathscr{D}'$, as $|\tau| \to \infty$ with $\frac{1}{2}\pi \geqslant \mathrm{ph}\,\tau \geqslant \frac{1}{2}\pi - \delta$. Remark that the difference between (A 27) and (A 23) is exponentially small when $0 \leqslant \mathrm{ph}\,\tau \leqslant \frac{1}{2}\pi - \delta$ and so (A 27) could be regarded as including (A 23) and covering the whole range $0 \leqslant \mathrm{ph}\,\tau \leqslant \frac{1}{2}\pi$.

The analogous formula for the derivative is

$$f'(\zeta\tau,\tau) \sim -3\pi^{\frac{1}{2}}\tau^{\frac{5}{6}}\zeta((1-\xi^2)/z)^{\frac{1}{4}}\{\mathrm{e}^{\frac{1}{4}\pi\tau}\,\mathrm{Ai}'\,(\tau^{\frac{2}{3}}z) + \mathrm{e}^{-\frac{5}{6}\pi\mathrm{i}-\frac{1}{4}\pi\tau}\,\mathrm{Ai}'\,(\tau^{\frac{2}{3}}z\,\mathrm{e}^{-\frac{2}{3}\pi\mathrm{i}})\} \qquad \text{(A 28)}$$

when $\mathrm{ph}\,\tau$ is near $\frac{1}{2}\pi$.

A similar procedure may be adopted for negative values of $\mathrm{ph}\,\tau$. The results, analogous to (A 27) and (A 28), are

$$f(\xi\tau,\tau) \sim 3\pi^{\frac{1}{2}}\tau^{\frac{7}{6}}\zeta(z/(1-\zeta^2))^{\frac{1}{4}}\{\exp[\tfrac{1}{4}\pi\tau]\,\mathrm{Ai}\,(\tau^{\frac{2}{3}}z) + \exp[\tfrac{1}{6}\pi\mathrm{i} - \tfrac{1}{4}\pi\tau]\,\mathrm{Ai}\,(\tau^{\frac{2}{3}}z\exp[\tfrac{2}{3}\pi\mathrm{i}])\} \quad \text{(A 29)}$$

and

$$f'(\zeta\tau,\tau) \sim -3\pi^{\frac{1}{2}}\tau^{\frac{5}{6}}\zeta((1-\zeta^2)/z)^{\frac{1}{4}}\{\exp[\tfrac{1}{4}\pi\tau]\,\mathrm{Ai}'\,(\tau^{\frac{2}{3}}z) + \exp[\tfrac{5}{6}\pi\mathrm{i} - \tfrac{1}{4}\pi\tau]\,\mathrm{Ai}'\,(\tau^{\frac{2}{3}}z\exp[\tfrac{2}{3}\pi\mathrm{i}])\} \quad \text{(A 30)}$$

for ζ in the mirror reflexion of $\mathscr{D}'$ in the real ζ-axis.

Comparable results for g may be derived by the same mechanism. It is found that

$$g(\zeta\tau,\tau) \sim -\tfrac{1}{3}f(\zeta\tau,\tau), \quad g'(\zeta\tau,\tau) \sim -\tfrac{1}{3}f'(\zeta\tau,\tau) \qquad \text{(A 31)}$$

for $|\mathrm{ph}\,\tau| \leqslant \frac{1}{2}\pi - \delta$. The analogues of (A 27) and (A 29) are

$$g(\zeta\tau,\tau) \sim -\pi^{\frac{1}{2}}\tau^{\frac{7}{6}}\zeta(z/(1-\zeta^2))^{\frac{1}{4}}\{\exp[\tfrac{1}{4}\pi\tau]\,\mathrm{Ai}\,(\tau^{\frac{2}{3}}z) - \exp[-\tfrac{1}{6}\pi\mathrm{i} - \tfrac{1}{4}\pi\tau]\,\mathrm{Ai}\,(\tau^{\frac{2}{3}}z\exp[-\tfrac{2}{3}\pi\mathrm{i}])\} \quad \text{(A 32)}$$

and

$$g(\zeta\tau,\tau) \sim -\pi^{\frac{1}{2}}\tau^{\frac{7}{6}}\zeta(z/(1-\zeta^2))^{\frac{1}{4}}\{\exp[\tfrac{1}{4}\pi\tau]\,\mathrm{Ai}\,(\tau^{\frac{2}{3}}z) - \exp[\tfrac{1}{6}\pi\mathrm{i} - \tfrac{1}{4}\pi\tau]\,\mathrm{Ai}\,(\tau^{\frac{2}{3}}z\exp[\tfrac{2}{3}\pi\mathrm{i}])\} \quad \text{(A 33)}$$

respectively.

Higher order approximations can be written down from (A 23)–(A 33) by replacing $\mathrm{Ai}\,(\tau^{\frac{2}{3}}z\,\mathrm{e}^{\frac{2}{3}j\pi\mathrm{i}})$ for $j = 0, \pm 1$ by

$$\mathrm{Ai}(\tau^{\frac{2}{3}}z\exp[\tfrac{2}{3}j\pi\mathrm{i}])\left\{1+\sum_{s=1}\frac{A_s(z)}{\tau^{2s}}\right\}+\frac{\exp[\frac{2}{3}j\pi\mathrm{i}]}{\tau^{\frac{4}{3}}}\,\mathrm{Ai}'(\tau^{\frac{2}{3}}z\exp[\tfrac{2}{3}j\pi\mathrm{i}])\sum_{s=}\frac{B_s(z)}{\tau^{2s}},$$

where A_s and B_s are defined in (A 20). In this connection it should be remarked that

$$B_0(z) = \frac{5}{16z^2}-\frac{z(2+3\zeta^2)}{4(1-\zeta^2)^3}. \tag{A 34}$$

On account of (A 31) an additional result is required to cope with the particular combination $f+3g$. It may be verified without difficulty that

$$g(\zeta\tau,\tau)+\tfrac{1}{3}f(\zeta\tau,\tau)\sim 2\pi^{\frac{1}{2}}\tau^{\frac{7}{6}}\exp[-\tfrac{1}{6}\pi\mathrm{i}-\tfrac{1}{4}\pi\tau]\,\zeta(z/(1-\zeta^2))^{\frac{1}{4}}\,\mathrm{Ai}\,(\tau^{\frac{2}{3}}z\exp[-\tfrac{2}{3}\pi\mathrm{i}]) \tag{A 35}$$

for $|\mathrm{ph}\,\tau| \leqslant \frac{1}{2}\pi-\delta$ with a corresponding relation for the derivative.

It is convenient to have available for various evaluations solutions of (A 1) other than f and g. Particular linear conbinations of f and g which play a useful rôle are defined by

$$f_1(\xi) = \frac{(-\frac{5}{2})!}{(-\frac{5}{4}-\frac{1}{4}\mathrm{i}\tau)!}f(\xi)+\frac{\frac{1}{2}!\,\tau^{\frac{3}{2}}\,\mathrm{e}^{-\frac{3}{4}\pi\mathrm{i}}}{(\frac{1}{4}-\frac{1}{4}\mathrm{i}\tau)!}g(\xi), \tag{A 36}$$

$$g_1(\xi) = \frac{(-\frac{5}{2})!}{(-\frac{5}{4}+\frac{1}{4}\mathrm{i}\tau)!}f(\xi)-\frac{\frac{1}{2}!\,\tau^{\frac{3}{2}}\,\mathrm{e}^{-\frac{1}{4}\pi\mathrm{i}}}{(\frac{1}{4}+\frac{1}{4}\mathrm{i}\tau)!}g(\xi), \tag{A 37}$$

with $|\mathrm{ph}\,\tau| < \pi$. The functions f_2 and g_2 are the same as f_1 and g_1 respectively with the signs of the second terms reversed.

These functions can be expressed as multiples of the confluent hypergeometric function Ψ. Understanding $\mathrm{ph}\,(\mathrm{i}\xi^2/\tau)$ to satisfy

$$\mathrm{ph}\,(\mathrm{i}\xi^2/\tau) = \tfrac{1}{2}\pi+2\,\mathrm{ph}\,\xi-\mathrm{ph}\,\tau$$

we have

$$f_1(\xi) = \xi^3\,\mathrm{e}^{-\mathrm{i}\xi^2/2\tau}\,\Psi(\tfrac{5}{4}-\tfrac{1}{4}\mathrm{i}\tau,\tfrac{5}{2};\,\mathrm{i}\xi^2/\tau), \tag{A 38}$$

$$g_1(\xi) = \xi^3\,\mathrm{e}^{\mathrm{i}\xi^2/2\tau}\,\Psi(\tfrac{5}{4}+\tfrac{1}{4}\mathrm{i}\tau,\tfrac{5}{2};\,(\mathrm{i}\xi^2/\tau)\,\mathrm{e}^{-\pi\mathrm{i}}), \tag{A 39}$$

$$f_2(\xi) = \xi^3\,\mathrm{e}^{-\mathrm{i}\xi^2/2\tau}\,\Psi(\tfrac{5}{4}-\tfrac{1}{4}\mathrm{i}\tau,\tfrac{5}{2};\,(\mathrm{i}\xi^2/\tau)\,\mathrm{e}^{2\pi\mathrm{i}}), \tag{A 40}$$

$$g_2(\xi) = \xi^3\,\mathrm{e}^{\mathrm{i}\xi^2/2\tau}\,\Psi(\tfrac{5}{4}+\tfrac{1}{4}\mathrm{i}\tau,\tfrac{5}{2};\,(\mathrm{i}\xi^2/\tau)\,\mathrm{e}^{\pi\mathrm{i}}). \tag{A 41}$$

Since

$$\Psi(a,c;\,z) = (1/z^a)\,\{1+O(1/|z|)\} \tag{A 42}$$

as $|z|\to\infty$ with $|\mathrm{ph}\,z| < \frac{3}{2}\pi$ it may be deduced that

$$f_1(\xi) \approx \xi^{\frac{1}{2}}\tau^{\frac{5}{4}}\exp[-\tfrac{5}{8}\pi\mathrm{i}-\mathrm{i}\xi^2/2\tau], \tag{A 43}$$

$$g_1(\xi) \approx \xi^{\frac{1}{2}}\tau^{\frac{5}{4}}\exp[\tfrac{5}{8}\pi\mathrm{i}+\mathrm{i}\xi^2/2\tau], \tag{A 44}$$

$$f_2(\xi) \approx \xi^{\frac{1}{2}}\tau^{\frac{5}{4}}\exp[-\tfrac{1}{8}\pi\mathrm{i}-\mathrm{i}\xi^2/2\tau], \tag{A 45}$$

$$g_2(\xi) \approx \xi^{\frac{1}{2}}\tau^{\frac{5}{4}}\exp[\tfrac{1}{8}\pi\mathrm{i}+\mathrm{i}\xi^2/2\tau] \tag{A 46}$$

as $\tau\to 0$ with ξ fixed.

Asymptotic results as $|\tau| \to \infty$ may be obtained by following the same route travelled for f and g. Omitting the details we quote, for $|\mathrm{ph}\,\tau| \leqslant \frac{1}{2}\pi$,

$$f_1(\zeta\tau, \tau) \sim 2\pi^{\frac{1}{2}}\tau^{\frac{23}{12}} \exp\left[-\tfrac{1}{8}\pi\tau - \tfrac{1}{4}\mathrm{i}\tau + \tfrac{1}{4}\mathrm{i}\tau \ln \tfrac{1}{4}\tau - \tfrac{13}{24}\pi\mathrm{i}\right] \zeta(z/(1-\zeta^2))^{\frac{1}{4}} \,\mathrm{Ai}\,(\tau^{\frac{2}{3}} z \exp[-\tfrac{2}{3}\pi\mathrm{i}]), \tag{A 47}$$

$$g_1(\zeta\tau, \tau) \sim 2\pi^{\frac{1}{2}}\tau^{\frac{23}{12}} \exp\left[-\tfrac{1}{8}\pi\tau + \tfrac{1}{4}\mathrm{i}\tau - \tfrac{1}{4}\mathrm{i}\tau \ln \tfrac{1}{4}\tau + \tfrac{13}{24}\pi\mathrm{i}\right] \zeta(z/(1-\zeta^2))^{\frac{1}{4}} \,\mathrm{Ai}\,(\tau^{\frac{2}{3}} z \exp[\tfrac{2}{3}\pi\mathrm{i}]), \tag{A 48}$$

$$f_2(\zeta\tau, \tau) \sim 2\pi^{\frac{1}{2}}\tau^{\frac{23}{12}} \exp\left[\tfrac{3}{8}\pi\tau - \tfrac{1}{4}\mathrm{i}\tau + \tfrac{1}{4}\mathrm{i}\tau \ln \tfrac{1}{4}\tau - \tfrac{3}{8}\pi\mathrm{i}\right] \zeta(z/(1-\zeta^2))^{\frac{1}{4}} \,\mathrm{Ai}\,(\tau^{\frac{2}{3}} z), \tag{A 49}$$

$$g_2(\zeta\tau, \tau) \sim 2\pi^{\frac{1}{2}}\tau^{\frac{23}{12}} \exp\left[\tfrac{3}{8}\pi\tau + \tfrac{1}{4}\mathrm{i}\tau - \tfrac{1}{4}\mathrm{i}\tau \ln \tfrac{1}{4}\tau + \tfrac{3}{8}\pi\mathrm{i}\right] \zeta(z/(1-\zeta^2))^{\frac{1}{4}} \,\mathrm{Ai}\,(\tau^{\frac{2}{3}} z) \tag{A 50}$$

and ζ in $\mathscr{D}'$.

The asymptotic formulae determined so far are not really suitable when $|\tau| \to \infty$ and ξ is fixed because then ζ tends to the origin which is a singular point of the mapping (A 16). A useful formula can be derived simply from the series expansion of f. Let $f(\xi) = \sum_{r=0} a_r \xi^{2r+3}$; the coefficients a_r satisfy the recurrence relation

$$2r(2r+3)\, a_r - a_{r-1} + a_{r-2}/\tau^2 = 0$$

subject to the initial conditions $a_0 = 1$, $a_1 = \frac{1}{10}$. Write

$$a_r = \frac{6(r+1)}{(2r+3)!} + c_r$$

so that c_r satisfies

$$2r(2r+3)\, c_r - c_{r-1} + \left\{\frac{6(r-1)}{(2r-3)!} + c_{r-2}\right\}\Big/\tau^2 = 0.$$

Assume that $c_s < 2/(s+1)!\,|\tau|^2$ for $s \leqslant r-1$. Then, for $|\tau| \geqslant 1$,

$$c_r < \frac{1}{4r(r+\frac{3}{2})\,|\tau|^2}\left\{\frac{6(r-1)}{(2r-3)!} + \frac{2}{(r-1)!} + \frac{2}{r!}\right\} < \frac{2}{(r+1)!\,|\tau|^2}$$

since $(2r-3)! \geqslant r!$ for $r \geqslant 3$. Since our assumption holds for $r = 2$, it is valid for any larger r by induction.

If therefore the expression for a_r is substituted in f we conclude that

$$f(\xi) = 3(\xi \cosh \xi - \sinh \xi) + O[\xi\{\exp(\xi^2) - 1\}/|\tau|^2] \tag{A 51}$$

for $|\tau| \geqslant 1$, the constant in the order term being independent of ξ and τ. Under the same conditions it may be demonstrated that

$$f'(\xi) = 3\xi \sinh \xi + O[\{(2\xi^2+1)\exp(\xi^2) - 1\}/|\tau|^2], \tag{A 52}$$

$$g(\xi) = \cosh \xi - \xi \sinh \xi + O[\xi^4 \exp(\xi^2)/|\tau|^2], \tag{A 53}$$

$$g'(\xi) = -\xi \cosh \xi + O[(2\xi^3 + \xi^5)\exp(\xi^2)/|\tau|^2]. \tag{A 54}$$

The next approximation stemming from the continuation of this device is

$$f(\xi) = 3\{(1 - \tfrac{5}{4}\tau^{-2})(\xi \cosh \xi - \sinh \xi) - (\xi^3/6\tau^2)(\xi \sinh \xi - \tfrac{5}{2}\cosh \xi)\}, \tag{A 55}$$

$$g(\xi) = \cosh \xi - \xi \sinh \xi + (\xi^3/6\tau^2)(\xi \cosh \xi - \tfrac{5}{2}\sinh \xi), \tag{A 56}$$

the errors being of $O(|\tau|^{-4})$.

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While (A 51)–(A 54) are satisfactory for some purposes they must be discarded when $|\xi|$ is large. If both $|\xi|$ and $|\tau|$ are large but $|\zeta|$ is small fresh approximations are a necessity. Now write $\kappa = \xi/\tau$ so that κ replaces ζ in (A 15). Change variables from κ to Z where

$$\frac{1}{Z^{\frac{1}{2}}}\frac{dZ}{d\kappa} = (1-\kappa^2)^{\frac{1}{2}}.$$

Thus

$$2Z^{\frac{1}{2}} = \tfrac{1}{2}\arcsin\kappa + \tfrac{1}{2}\kappa(1-\kappa^2)^{\frac{1}{2}}. \tag{A 57}$$

Define $(1-\kappa^2)^{\frac{1}{2}}$ in the κ-plane in the same way that v is defined in the u-plane. Take $\arcsin\kappa$ to lie between 0 and $\frac{1}{2}\pi$ when $0<\kappa<1$ and by continuity elsewhere. Then $Z^{\frac{1}{2}}$ is positive for $0<\kappa<1$ and the transformation supplies a one-to-one mapping from $0<\operatorname{Re}\kappa<1$ to the interior of $|Z| < \frac{1}{16}\pi^2$ with a cut along the negative real Z-axis. The mapping is, indeed, one-to-one over a bigger region but that described is sufficient for our purposes.

Substitute in the differential equation

$$w = \hat{W}/Z^{\frac{1}{4}}(1-\kappa^2)^{\frac{1}{4}}.$$

The differential equation for $\hat{W}$ is

$$\frac{d^2\hat{W}}{dZ^2} = \left\{\frac{\tau^2}{Z} + \frac{5}{16Z^2} + h_1(Z)\right\}\hat{W},$$

where

$$Zh_1(Z) = \frac{2}{\kappa^2} - \frac{1}{2Z} + \frac{2}{1-\kappa^2} - \frac{1}{2(1-\kappa^2)^2} - \frac{5\kappa^2}{4(1-\kappa^2)^3}.$$

As $\tau\to\infty$ through real positive values, one may apply similar reasoning to that followed in deriving (A 23). Consequently,

$$f(\kappa\tau,\tau) \sim \frac{3\pi^{\frac{1}{2}}\tau^{\frac{3}{2}}\kappa Z^{\frac{1}{4}}}{(1-\kappa^2)^{\frac{1}{4}}} I_{\frac{3}{2}}(2\tau Z^{\frac{1}{2}})$$

$$\sim \frac{3\kappa}{2Z^{\frac{1}{2}}(1-\kappa^2)^{\frac{1}{4}}}\{2\tau Z^{\frac{1}{2}}\cosh(2\tau Z^{\frac{1}{2}}) - \sinh(2\tau Z^{\frac{1}{2}})\} \tag{A 58}$$

and

$$g(\kappa\tau,\tau) \sim \frac{\kappa}{2Z^{\frac{1}{2}}(1-\kappa^2)^{\frac{1}{4}}}\{\cosh(2\tau Z^{\frac{1}{2}}) - 2\tau Z^{\frac{1}{2}}\sinh(2\tau Z^{\frac{1}{2}})\} \tag{A 59}$$

as $\tau\to\infty$ with $|Z| < \frac{1}{16}\pi^2$, $0<\operatorname{Re}\kappa<1$. In particular

$$g(\kappa\tau,\tau) + \tfrac{1}{3}f(\kappa\tau,\tau) \sim \frac{\kappa(1+2\tau Z^{\frac{1}{2}})}{2Z^{\frac{1}{2}}(1-\kappa^2)^{\frac{1}{4}}}\exp[-2\tau Z^{\frac{1}{2}}]. \tag{A 60}$$

Appendix B

The enumeration of the zeros of

$$\Delta = \mu\{uf'(s) + ivf(s)\} - \lambda\{ug'(s) + ivg(s)\},$$

where

$$\lambda = uf'(\xi) - iwf(\xi), \quad \mu = ug'(\xi) - iwg(\xi)$$

will be carried out by examining the variation in the phase of Δ as u tranverses the closed curve consisting of the real axis above the branch lines and a large semi-circle in the upper half of the u-plane. The consideration will be split into several portions, each of which will be dealt with separately. We start with the real axis.

B 1. *The interval* $1 \leqslant u \leqslant 1/M$

Recalling that $s = kh/M$, $\xi = s - khu$ we see that $0 \leqslant \xi < s$ on $1 \leqslant u \leqslant 1/M$. Also $\tau = us$ is positive and $\xi < \tau$, $s \leqslant \tau$. Both v and w are negative imaginary on the interval and $v = -iu\eta_0$, $w = -iu\eta$ where the positive quantities η_0 and η are defined by

$$\eta_0 = (1 - s^2/\tau^2)^{\frac{1}{2}}, \quad \eta = (1 - \xi^2/\tau^2)^{\frac{1}{2}}.$$

In view of these results

$$\begin{aligned} \Delta/u^2 = {} & g'(\xi) f'(s) - f'(\xi) g'(s) + \eta_0\{g'(\xi) f(s) - f'(\xi) g(s)\} \\ & - \eta\{g(\xi) f'(s) - f(\xi) g'(s)\} - \eta\eta_0\{g(\xi) f(s) - f(\xi) g(s)\}. \end{aligned} \tag{B 1}$$

It is known from appendix A that both f and f' are positive for $0 < \xi \leqslant \tau$. Also, from (A 8),

$$\{g(\xi)/f(\xi)\}' = -3\xi^2/f^2.$$

Hence g/f is a decreasing function of ξ and so, since both $f(\xi)$ and $f(s)$ are positive,

$$g(\xi) f(s) - f(\xi) g(s) > 0 \tag{B 2}$$

for $0 \leqslant \xi < s \leqslant \tau$. Equally well, $f'(s)\, g/f - g'(s)$ is a decreasing function of ξ which, by (A 8), is positive at $\xi = s$. Hence

$$g(\xi) f'(s) - f(\xi) g'(s) > 0 \tag{B 3}$$

for $0 \leqslant \xi < s \leqslant \tau$.

Again, (A 8) and the differential equation (A 1) imply that

$$\{g'(\xi)/f'(\xi)\}' = 3\xi^2\eta^2/f'^2.$$

Therefore g'/f' is an increasing function of ξ and

$$g'(\xi) f'(s) - f'(\xi) g'(s) < 0 \tag{B 4}$$

for $0 < \xi < s \leqslant \tau$. Moreover, $f(s)\, g'/f' - g(s)$ is an increasing function of ξ which is negative at $\xi = s$. Consequently

$$g'(\xi) f(s) - f'(\xi) g(s) < 0 \tag{B 5}$$

for $0 < \xi < s \leqslant \tau$.

The combination of (B 1)–(B 5) indicates that Δ cannot be positive on $1 \leqslant u \leqslant 1/M$. It can be zero only if each of the four groups of terms in (B 1) vanishes. But that is clearly impossible by virtue of (B 3) and $\eta \neq 0$. Hence Δ is negative on the interval under consideration.

B 2. *The interval* $0 < \epsilon \leqslant u \leqslant 1$

On the interval $0 < u < 1$ at least one of v and w is positive. In addition, $s > \tau$ so that the inequalities (B 2)–(B 5) are not available because change of sign by $f(s)$ and $f'(s)$ cannot be discounted. A different approach from that of § B 1 must therefore be adopted.

First, observe that

$$\{f'(\xi) - \eta f(\xi)\}' = \frac{2}{\xi} f'(\xi) + \frac{\xi}{\tau^2\eta} f(\xi) - \eta\{f'(\xi) - \eta f(\xi)\}.$$

If there is a point ξ satisfying $0 < \xi \leqslant \tau$, where $f' - \eta f$ is zero, this relation shows that its derivative is positive there. Thus, $f' - \eta f$ must pass through a zero by increasing from negative to positive values. However, $f' - \eta f$ goes positive as ξ increases from zero. Hence

$$f'(\xi) - \eta f(\xi) > 0 \tag{B 6}$$

for $0 < \xi \leqslant \tau$.

Now write
$$\Delta = \lambda\{uf'(s)+\mathrm{i}vf(s)\}\Delta_0.$$
Then
$$\Delta_0 = \frac{\mu}{\lambda} - \frac{ug'(s)+\mathrm{i}vg(s)}{uf'(s)+\mathrm{i}vf(s)}.$$

On $1/(M+1) \leqslant u \leqslant 1$, v is positive and w is negative imaginary. Therefore μ/λ is real and Δ_0 has negative imaginary part on account of the Wronskian relation (A 8). Indeed, the imaginary part can be zero only at $u = 1$.

Similarly, on $\epsilon \leqslant u \leqslant 1/(M+1)$, Δ_0 always has a negative imaginary part. It is never zero except maybe at $u = 0$.

From § B 1, Δ_0 is negative at $u = 1$ and so we can regard the phase of Δ_0 as lying between 0 and $-\pi$ for $\epsilon \leqslant u \leqslant 1$. The phase of Δ may now be deduced when the behaviour of $\lambda(uf'+\mathrm{i}vf)$ is known on the interval, allowance being made for possible changes of sign in f and f'.

Let the zeros of $f(\xi)$ in $\xi > \tau$ be denoted by $\xi_1(\tau), \xi_2(\tau), \ldots$. It is assumed that they are arranged in ascending order so that $\xi_1 < \xi_2 < \ldots$. As τ decreases, Sturm–Liouville theory indicates that the new positions of the zeros must interlace the old and their number increase. A double zero can never occur because the Wronskian relation prevents f and f' vanishing simultaneously. Further, $f(\xi)$ is positive at $\xi = \tau$. Consequently, the only possibility is that $\xi_j(\tau)$ decreases as τ decreases.

The zeros of $f(s)$ therefore arise at a decreasing sequence of values of τ whose members τ_j are given by $\xi_1(\tau_1) = s, \xi_2(\tau_2) = s, \ldots$. Once $\xi_j(\tau)$ has passed through s it remains to the left of s for all smaller values of τ. For such a ξ_j there must be a value of τ in $(0, 1)$ such that $\xi_j(\tau) = s-M\tau$ because $s-M\tau$ increases from $s(1-M)$ to s as τ decreases from 1 to 0. It therefore follows that to every zero that $f(s)$ goes through as τ decreases from 1 to 0 there corresponds a zero of $f(s-M\tau)$ as τ decreases from $1/(M+1)$ to zero. The same statement can obviously be made about the derivatives. Consequently, for every complete circuit of the origin made by the phase of $uf'(s)+\mathrm{i}vf(s)$ as τ moves from 0 to 1 there is an opposite circuit made by the phase of $uf'(\xi)-\mathrm{i}wf(\xi)$ as τ travels from 0 to $1/(M+1)$. By (B6) $uf'(\xi)-\mathrm{i}wf(\xi)$ is positive for τ between $1/(M+1)$ and 1. Therefore, the conclusion is unmodified if the interval for τ is taken as $(0, 1)$ in both cases.

Combining this information with that concerning Δ_0 we see that, for τ going from 0 to 1, the phase of Δ changes from its value near the origin to negative real by a route which is essentially clockwise in its general trend and whose angular variation is less than 2π.

B 3. *The interval* $-1/(1-M) \leqslant u \leqslant -\epsilon$

For the interval under consideration in this section u is negative so τ is replaced by $-|\tau|$. Advantage is then taken of (A 5) to replace $-|\tau|$ in the second argument of f and g by $|\tau|$.

When $u < -1$, v is positive imaginary since points just above the real axis are being investigated. Thus $v = \mathrm{i}\,|u|\,\eta_0$ and
$$uf'(s)+\mathrm{i}vf(s) = -|u|\,\{f'(s)+\eta_0 f(s)\}$$
which is negative since $s < |\tau|$.

The argument of § B 2 is now repeated. In this case Δ_0 has a positive imaginary part. The main alteration is to start with the zeros of $f(s+M|\tau|)$. Any ξ_j which passes to the left of $s+M|\tau|$ as $|\tau|$ goes from $1/(1-M)$ to 0 must eventually give rise to a zero of $f(s)$ as $|\tau|$ decreases from 1 to 0.

Accordingly, as τ moves from $-1/(1-M)$ to 0, the phase of Δ alters from negative real to its value near the origin by a clockwise path of angular extent less than 2π.

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B 4. *The interval* $u \leqslant -1/(1-M)$

The device described in the first paragraph of § B 3 is first followed. Then, since both v and w are positive imaginary,

$$\frac{\Delta}{u^2\{f'(s)+\eta_0 f(s)\}} = g'(\xi)-\eta g(\xi)-\frac{g'(s)+\eta_0 g(s)}{f'(s)+\eta_0 f(s)}\{f'(\xi)-\eta f(\xi)\}. \tag{B 7}$$

Since $s < |\tau|$, the denominator is positive and, since $\xi \leqslant |\tau|$, the last factor of the last term of (B 7) is positive by (B 6). Therefore, by (B 10) below, the right-hand side of (B 7) increases as s increases. It is negative at $s = \xi$ because of the value of the Wronskian. Since $s < \xi$ on the intervals, the inference is that Δ is negative.

B 5. *The neighbourhood of the origin*

Near the origin $|u|$ and $|\tau|$ are small while ξ has effectively the constant value s. In these circumstances we can avail ourselves of (A 11)–(A 14) and (A 43)–(A 46).

If $0 \leqslant \mathrm{ph}\, u \leqslant \frac{1}{2}\pi$, express Δ in terms of f_1 and g_1 by means of (A 36) and (A 37). Then

$$\Delta = \tfrac{3}{2}\Delta_1 \tau^{-\frac{3}{2}} \mathrm{e}^{-\frac{1}{4}\pi\tau},$$

where Δ_1 is the same as Δ with f_1 and g_1 in place of f and g respectively. Now (A 43) and (A 44) give

$$\Delta \approx -6\mathrm{i}s^2 u\, \mathrm{e}^{\mathrm{i}sM} \tag{B 8}$$

for small $|u|$.

When $\frac{1}{2}\pi \leqslant \mathrm{ph}\, u \leqslant \pi$, substitute for f and g in terms of f_1 and g_2. Then (A 43) and (A 46) show that (B 8) is still valid.

It is important to recognize that (B 8) conveys the information that the essential singularities of f and g near $\tau = 0$ cancel in Δ so that special treatment to account for them is unnecessary.

From (B 8), Δ has a simple zero at the origin and its phase decreases by π as the origin is encircled from $\mathrm{ph}\, u = \pi$ to $\mathrm{ph}\, u = 0$.

B 6. *The interval* $u \leqslant 1/M$

At this stage the opportunity is taken to unite the results of §§ B 1–B 5 into a single statement governing the phase of Δ on the real axis.

As u increases from $-\infty$ to $1/M$, passing above the origin, the phase of Δ proceeds from negative real values to negative real values by making a 2π circuit of the origin in the clockwise sense.

B 7. *The interval* $1/M < u \leqslant 2/M$

Up to this point the only real zero of Δ which has occurred has been at the origin and the net phase variations of Δ have not been very great. Unfortunately, there are zeros of Δ in $u > 1/M$ as will be discovered subsequently and so the analysis for this interval is much more complicated than for the previous ones.

On $u > 1/M$, ξ is negative and may be written as $-\xi_1$ where $\xi_1 = M\tau - s$. Clearly $0 \leqslant \xi_1 < \tau$ and $0 < s < \tau$. Both v and w are negative imaginary. Drawing benefit from (A 4), we have

$$\lambda = u\{f'(\xi_1)+\eta_1 f(\xi_1)\}$$

where η_1 is the same as η with ξ_1 for ξ. Thus λ is positive and so is $uf'(s)+\mathrm{i}vf(s)$. Hence Δ has the same sign as Δ_0 which now takes the form given by

$$\Delta_0 = -\frac{g'(\xi_1)+\eta_1 g(\xi_1)}{f'(\xi_1)+\eta_1 f(\xi_1)} - \frac{g'(s)+\eta_0 g(s)}{f'(s)+\eta_0 f(s)}.$$

Put
$$F(\xi,\tau)=\frac{g'(\xi)+\eta g(\xi)}{f'(\xi)+\eta f(\xi)}.$$

Then
$$\Delta_0=-F(\xi_1,\tau)-F(s,\tau). \tag{B 9}$$

The first task is to show that if, for some τ, $g'(\xi,\tau)$ changes sign as ξ goes from 0 to τ then Δ_0 cannot vanish for this value of τ. From (A 1) and (A 8)

$$\frac{\partial}{\partial\xi}F(\xi,\tau)=\frac{-3\xi(2-\xi^2/\tau^2)}{\eta\{f'(\xi)+\eta f(\xi)\}^2}. \tag{B 10}$$

Consequently, F decreases steadily as ξ increases from 0 to τ. Therefore, F must be positive for $0<\xi\leqslant\tau$ if it is positive at $\xi=\tau$. In particular, this must be true at $\xi=\xi_1$ and $\xi=s$ so that Δ_0 is negative. However, g' is negative near $\xi=0$ and if it changes sign, which can happen only once, it must be positive at $\xi=s$ and so must F. Thus, in locating any zeros of Δ_0, attention can be restricted to the situation where $g'(\xi,\tau)$ remains negative for $0<\xi\leqslant\tau$.

When $u\leqslant 2/M$, $s\geqslant\frac{1}{2}M\tau$ and $\xi_1\leqslant s$. Therefore, if $F(\xi_1,\tau)$ is negative, (B 10) implies that $F(s,\tau)$ is negative and Δ_0 cannot be zero. Thus points at which $F(\xi_1,\tau)$ is negative can be excluded and only the feasibility of Δ_0 possessing a zero at which $F(\xi_1,\tau)$ is positive need be scrutinized. Necessarily, $F(s,\tau)$ will have to be negative for a zero to be on the cards.

The target of the next piece of analysis to demonstrate that Δ_0 must be increasing at such a zero.

Let $\partial/\partial\tau$ denote a derivative with respect to τ in which ξ is kept constant. Then $\partial f(\xi)/\partial\tau$ satisfies

$$\left(\frac{\partial f}{\partial\tau}\right)''-\frac{2}{\xi}\left(\frac{\partial f}{\partial\tau}\right)'-\left(1-\frac{\xi^2}{\tau^2}\right)\frac{\partial f}{\partial\tau}=\frac{2\xi^2}{\tau^3}f.$$

Variation of parameters, together with (A 2) and (A 3), supplies

$$\frac{\partial f(\xi)}{\partial\tau}=\frac{2}{3\tau^3}\int_0^\xi f(t)\{f(\xi)g(t)-g(\xi)f(t)\}\,\mathrm{d}t, \tag{B 11}$$

whence
$$\frac{\partial}{\partial\tau}f'(\xi)=\frac{2}{3\tau^3}\int_0^\xi f(t)\{f'(\xi)g(t)-g'(\xi)f(t)\}\,\mathrm{d}t. \tag{B 12}$$

Similarly
$$\frac{\partial}{\partial\tau}g(\xi)=\frac{2}{3\tau^3}\int_0^\xi g(t)\{f(\xi)g(t)-g(\xi)f(t)\}\,\mathrm{d}t, \tag{B 13}$$

$$\frac{\partial}{\partial\tau}g'(\xi)=\frac{2}{3\tau^3}\int_0^\xi g(t)\{f'(\xi)g(t)-g'(\xi)f(t)\}\,\mathrm{d}t. \tag{B 14}$$

It follows that
$$\frac{\partial}{\partial\tau}F(\xi,\tau)=\frac{3\xi^4/\tau^3\eta}{\{f'(\xi)+\eta f(\xi)\}^2}+\frac{2}{3\tau^3}\int_0^\xi\{g(t)-F(\xi,\tau)f(t)\}^2\,\mathrm{d}t. \tag{B 15}$$

The alteration in Δ_0 as u varies can now be calculated. It is given in essentials by

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\{F(\xi,\tau)+F(s,\tau)\}=M\frac{\partial}{\partial\xi}F(\xi,\tau)+\frac{\partial}{\partial\tau}F(\xi,\tau)+\frac{\partial}{\partial\tau}F(s,\tau) \tag{B 16}$$

when $\xi=\xi_1$, after substituting from (B 10) and (B 15).

Dropping the explicit mention of τ in F, we know from (B10) that $F(\xi)\geqslant F(\xi_1)$ for $\xi\leqslant\xi_1$. Hence
$$g'(\xi)-F(\xi_1)f'(\xi)+\eta\{g(\xi)-F(\xi_1)f(\xi)\}\geqslant 0 \tag{B 17}$$

for $0 \leqslant \xi \leqslant \xi_1$. The first two terms of (B 17) furnish a negative contribution and so $g(\xi) > F(\xi_1) f(\xi)$ for $0 \leqslant \xi \leqslant \xi_1$. Accordingly, it is legitimate to divide by $g(\xi) - F(\xi_1) f(\xi)$. It may therefore be asserted that

$$\frac{d}{d\xi}\left[\frac{g'(\xi)-F(\xi_1)f'(\xi)}{\xi^2\{g(\xi)-F(\xi_1)f(\xi)\}}+\frac{1}{\xi}\right] = -\frac{1}{\tau^2}-\frac{1}{\xi^2}\left\{\frac{g'(\xi)-F(\xi_1)f'(\xi)}{g(\xi)-F(\xi_1)f(\xi)}\right\}^2 < -\frac{1}{\tau^2},$$

whence
$$g'(\xi)-F(\xi_1)f'(\xi)+\{\xi+3\xi^2F(\xi_1)+\xi^3/\tau^2\}\{g(\xi)-F(\xi_1)f(\xi)\} \leqslant 0 \tag{B 18}$$
for $0 \leqslant \xi \leqslant \xi_1$.

Putting $\xi = \xi_1$ in (B 17) and (B 18) makes evident that

$$\xi_1 + 3\xi_1^2 F(\xi_1) + \xi_1^3/\tau^2 \leqslant \eta_1 \tag{B 19}$$

is a necessary condition for the existence of a zero of Δ_0 in which $F(\xi_1) \geqslant 0$. Obviously, no zero of this type can arise for those values of u which satisfy $1 \leqslant M\tau - s \leqslant s$.

To assess the relative magnitudes of the terms in (B 16) when a zero is possible further inequalities are needed. Now

$$\frac{d}{d\xi}\left[\frac{1}{\xi^2}\{g'(\xi)-F(\xi_1)f'(\xi)\}+\frac{1}{\xi}\right] = \left(\frac{1}{\xi^2}-\frac{1}{\tau^2}\right)\{g(\xi)-F(\xi_1)f(\xi)\}-\frac{1}{\xi^2}.$$

Because $g(\xi) - F(\xi_1) f(\xi)$ is positive and steadily falls, as ξ increases, from the value of unity at $\xi = 0$, the terms involving $1/\xi^2$ on the right-hand side must be negative. Therefore

$$g'(\xi)-F(\xi_1)f'(\xi)+\xi+3F(\xi_1)\,\xi^2+\frac{\xi^2}{\tau^2}\int_0^{\xi}\{g(t)-F(\xi_1)f(t)\}\,dt \leqslant 0 \tag{B 20}$$

for $0 \leqslant \xi \leqslant \xi_1$.

Consider next $\quad G(\xi) = -\tau^2\{g'(\xi)-F(\xi_1)f'(\xi)\}\{g(\xi)-F(\xi_1)f(\xi)\}/\xi^2.$

By virtue of (B 17) and (B 20), $G(\xi)$ exceeds $\tau^2/\xi\eta$ which has a minimum of 2τ when $\xi = \tau/\sqrt{2}$. Consequently, $G(\xi) > 1$ for $\tau > \frac{1}{2}$. On the other hand, since $\eta \leqslant 1$, (B 17) implies that $g(\xi) - F(\xi_1) f(\xi) \geqslant e^{-\xi}$ and so, from (B 20),

$$g'(\xi)-F(\xi_1)f'(\xi)+\xi+3\xi^3/4\tau^2 \leqslant 0$$

for $0 \leqslant \xi \leqslant \tau \leqslant \frac{1}{2}$. Hence $\quad G(\xi) > \tau^2(1+3\xi^2/4\tau^2)^2/\xi\eta$

and the right-hand side has a minimum when $12\xi^2/\tau^2 = 17-\sqrt{193}$. This minimum is not less than 3τ and so it has been established that $G(\xi) \geqslant 1$ for $\tau \geqslant \frac{1}{3}$. Moreover, a deduction from (B 17) is

$$g(\xi)-F(\xi_1)f(\xi) > e^{-z_1},$$

where
$$z_1 = \tfrac{1}{2}\xi\eta + \tfrac{1}{2}\tau \arcsin(\xi/\tau). \tag{B 21}$$

The inequalities (B 18) and (B 21) entail

$$G(\xi) \geqslant (1+\tau^2/\xi^2)\,e^{-2z_1}.$$

The right-hand side diminishes as ξ grows and must therefore have $2\,e^{-\frac{1}{2}\pi\tau}$ as a lower bound for $\xi \leqslant \tau$. This lower bound is not less than unity if $\tau < (2/\pi)\ln 2 \approx 0.4$. Consequently, we have proved that

$$G(\xi) \geqslant 1 \tag{B 22}$$

for $0 \leqslant \xi \leqslant \xi_1$ without any restriction on τ. It is an immediate consequence of applying (A 8) to (B 22) that

$$\{f'(\xi_1)+\eta_1 f(\xi_1)\}^2 \leqslant 9\xi_1\eta_1\tau^2 \tag{B 23}$$

when $F(\xi_1) \geqslant 0$.

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Amalgamating (B 10), (B 15) and (B 23) we obtain

$$M\frac{\partial}{\partial\xi}F(\xi,\tau)+\frac{\partial F}{\partial\tau}\leqslant\frac{3\xi(2-\xi^2/\tau^2)\,(\xi/\tau-M)}{\eta\{f'(\xi)+\eta f(\xi)\}^2},\tag{B 24}$$

when $F(\xi)\geqslant 0$. Since $\xi_1\leqslant M\tau$, a significant upshot of the analysis so far is that, where $F(\xi_1,\tau)$ is non-negative, $F(\xi_1,\tau)$ has a descending value as u increases from $1/M$.

In order to take advantage of (B 24) in (B 16), we first compare values of the denominator of (B 24) with those at $\xi = s$. Now

$$\frac{d}{d\xi}\left[\{f'(\xi)+\eta f(\xi)\}\frac{1}{\xi^2\eta^{\frac{1}{2}}}\right]=\frac{1}{2\xi\tau^2\eta^{\frac{5}{2}}}\{f'(\xi)-\eta f(\xi)\}+\frac{\eta^{\frac{1}{2}}}{\xi^2}\left\{f'(\xi)-\frac{2}{\xi}f(\xi)\right\}+\frac{\eta^{\frac{3}{2}}}{\xi^2}f(\xi).$$

On account of (B 6) and (A 9) the right-hand side is positive and so

$$\{f'(s)+\eta_0 f(s)\}\,\xi^2\eta^{\frac{1}{2}}\geqslant\{f'(\xi)+\eta f(\xi)\}\,s^2\eta_0^{\frac{1}{2}}\tag{B 25}$$

for $0\leqslant\xi\leqslant s$.

The function $g(\xi)-F(s)f(\xi)$ can change sign at most once as ξ goes from 0 to s. In fact, no change of sign arises because the function is positive at both $\xi = 0$ and $\xi = s$. Similarly $g'(\xi)-F(s)f'(\xi)$ is never positive, i.e.

$$1\geqslant g(\xi)-F(s)f(\xi)>0,\quad g'(\xi)-F(s)f'(\xi)\leqslant 0\tag{B 26}$$

for $0\leqslant\xi\leqslant s$.

Combining (B 15), (B 23) – (B 26) and remembering that $\xi_1+s = M\tau$, we obtain

$$\frac{d}{d\tau}\{F(\xi_1)+F(s)\}\leqslant\frac{3\xi_1^3(\xi_1-s)\,\{1-(\xi_1^2+\xi_1 s+s^2)/\tau^2\}}{\tau^3\eta_1\eta_0^2\{f'(\xi_1)+\eta_1 f(\xi_1)\}^2}\tag{B 27}$$

when $F(\xi_1)\geqslant 0$. But $\quad \xi_1^2+\xi_1 s+s^2=\xi_1^2-M\tau\xi_1+M^2\tau^2\leqslant\tau^2$

because $\xi_1\leqslant M\tau$ and $M\leqslant 1$. The right-hand side of (B 27) cannot, therefore be positive.

The significance of this result is that Δ_0 must steadily increase with u in any interval where $\xi_1\leqslant s$ and $F(\xi_1)\geqslant 0$. Furthermore, once $F(\xi_1)$ has gone negative (B 24) prevents it becoming positive again for any greater u; then (B 10) forces $F(s)$ to be negative for $\xi_1\leqslant s$. In other words, either Δ_0 increases through a zero and remains positive until $u = 2/M$ or it stays negative throughout $1/M\leqslant u\leqslant 2/M$.

In all circumstances Δ_0 has at most one zero on $1/M\leqslant u\leqslant 2/M$. There is no zero if $F(s, 2s/M)>0$ and exactly one zero if $F(s, 2s/M)\leqslant 0$, the zero being at $u = 2/M$ if the equality holds.

B 8. *The interval* $u > 2/M$

In this interval $\xi_1 > s$ so that, if $F(\xi_1)$ is positive, (B 10) implies that $F(s)>0$ and Δ_0 cannot be zero. Only the case where $F(\xi_1)$ is negative therefore needs to be considered. However, if $F(\xi_1)<0$, the derivation of (B 23) fails and so it may no longer be true that Δ_0 is increasing at a zero. A proof by other means that the left-hand side of (B 27) was negative would settle the matter. Regrettably, this avenue is not open because it is false to say that the left-hand side of (B 27) is negative at a zero of Δ_0. However, what can be shown is that once the derivative has become positive it takes positive values for succeeding values of u and this will be sufficient for our purposes.

Denote the operator $M\partial/\partial\xi+\partial/\partial\tau$ by D. The behaviour of the derivative at points in $\xi > s$ where $D\{F(\xi)+F(s)\}\leqslant 0$ will now be discussed. At such points $DF(\xi)<0$ because $DF(s)>0$ by (B 15).

Our aim is to calculate the second derivative from (B 10) and (B 15), and then estimate its value. Now

$$\begin{aligned} \mathrm{D}\int_0^\xi \{g(t)-F(\xi)f(t)\}^2\,\mathrm{d}t = {} & M\{g(\xi)-F(\xi)f(\xi)\}^2 \\ & -2\int_0^\xi \{g(t)-F(\xi)f(t)\}f(t)\,\mathrm{D}F(\xi)\,\mathrm{d}t \\ & +2\int_0^\xi \{g(t)-F(\xi)f(t)\}\left\{\frac{\partial}{\partial\tau}g(t)-F(\xi)\frac{\partial}{\partial\tau}f(t)\right\}\mathrm{d}t. \end{aligned} \qquad \text{(B 28)}$$

From (B 11) and (B 13)

$$\frac{\partial}{\partial\tau}g(t)-F(\xi)\frac{\partial}{\partial\tau}f(t) = \frac{2}{3\tau^3}\int_0^t \{g(u)-F(\xi)f(u)\}\{f(t)\,g(u)-g(t)f(u)\}\,\mathrm{d}u > 0 \qquad \text{(B 29)}$$

for $0 < t \leqslant \xi$ by (B 2) and (B 26). Applying (B 26) and (B 29) to (B 28) we obtain

$$\mathrm{D}\int_0^\xi \{g(t)-F(\xi)f(t)\}^2\,\mathrm{d}t > M\{g(\xi)-F(\xi)f(\xi)\}^2-2\int_0^\xi \{g(t)-F(\xi)f(t)\}f(t)\,\mathrm{D}F(\xi)\,\mathrm{d}t. \qquad \text{(B 30)}$$

There is a similar result with s for ξ, the first term on the right-hand side being missing.

On account of (B 26) and the positivity of f,

$$\begin{aligned} \int_0^\xi \{g(t)-F(\xi)f(t)\}f(t)\,\mathrm{d}t & \geqslant \int_0^s \{g(t)-F(\xi)f(t)\}f(t)\,\mathrm{d}t \\ & \geqslant \int_0^s \{g(t)-F(s)f(t)\}f(t)\,\mathrm{d}t \end{aligned}$$

since $F(\xi) \leqslant F(s)$ for $\xi \geqslant s$. This may be substituted in (B 30) since $\mathrm{D}F(\xi) < 0$. Combining this with the (B 30) in terms of s and noting that $\mathrm{D}\{F(\xi)+F(s)\} \leqslant 0$, we have

$$\mathrm{D}\int_0^\xi \{g(t)-F(\xi)f(t)\}^2\,\mathrm{d}t+\mathrm{D}\int_0^s \{g(t)-F(s)f(t)\}^2\,\mathrm{d}t > M\{g(\xi)-F(\xi)f(\xi)\}^2. \qquad \text{(B 31)}$$

Further

$$\begin{aligned} \mathrm{D}\frac{1}{\eta}\{g(\xi)-F(\xi)f(\xi)\}^2 = {} & \{g(\xi)-F(\xi)f(\xi)\}^2\left\{\frac{\xi}{\tau^3\eta^3}(M\tau-\xi)-2M\right\} \\ & +\frac{2}{\eta}\{g(\xi)-F(\xi)f(\xi)\}\left\{\frac{\partial}{\partial\tau}g(\xi)-F(\xi)\frac{\partial}{\partial\tau}f(\xi)-f(\xi)\,\mathrm{D}F(\xi)\right\} \\ > {} & \{g(\xi)-F(\xi)f(\xi)\}^2\left\{\frac{\xi}{\tau^3\eta^3}(M\tau-\xi)-2M\right\} \\ & -\frac{2}{\eta}\{g(\xi)-F(\xi)f(\xi)\}f(\xi)\,\mathrm{D}F(\xi) \end{aligned}$$

by (B 26) and (B 29). By taking the derivative with respect to t of

$$\{g(t)-F(\xi)f(t)\}f(t)\,(1-t^2/\tau^2)^{-\frac{1}{2}}$$

we can see, from (B 17), (B 6) and (B 26), that it is an increasing function of t for $t \leqslant \xi$. Hence

$$\begin{aligned} \frac{1}{\eta}\{g(\xi)-F(\xi)f(\xi)\}f(\xi) & \geqslant \frac{1}{\eta_0}\{g(s)-F(\xi)f(s)\}f(s) \\ & \geqslant \frac{1}{\eta_0}\{g(s)-F(s)f(s)\}f(s) \end{aligned}$$

for $\xi \geqslant s$. Therefore

$$\mathrm{D}\frac{1}{\eta}\{g(\xi)-F(\xi)f(\xi)\}^2+\mathrm{D}\frac{1}{\eta_0}\{g(s)-F(s)f(s)\}^2$$
$$> \{g(\xi)-F(\xi)f(\xi)\}^2\left\{\frac{\xi}{\tau^3\eta^3}(M\tau-\xi)-2M\right\}-\frac{s^2}{\tau^3\eta_0^3}\{g(s)-F(s)f(s)\}^2. \quad \text{(B 32)}$$

Since $$\{f'(\xi)+\eta f(\xi)\}^{-1} = \tfrac{1}{3}\xi^{-2}\{g(\xi)-F(\xi)f(\xi)\}$$

we may conclude from (B 31) and (B 32) that

$$\mathrm{D}[3\tau^3\mathrm{D}\{F(\xi)+F(s)\}] > \frac{M}{\tau^2\eta^3}\left(\frac{\xi}{M\tau}+2-\frac{9\tau^2}{\xi^2}+\frac{6\tau^4}{\xi^4}\right)(M\tau-\xi)\{g(\xi)-F(\xi)f(\xi)\}^2$$
$$-\frac{s^2}{\tau^3\eta_0^3}\{g(s)-F(s)f(s)\}^2$$
$$-\frac{2M\tau^3}{\eta\xi^3}\left(2-\frac{\xi^2}{\tau^2}\right)\{g(\xi)-F(\xi)f(\xi)\}\,\mathrm{D}\{g(\xi)-F(\xi)f(\xi)\} \quad \text{(B 33)}$$

for $\xi \geqslant s$ and $\mathrm{D}\{F(\xi)+F(s)\} \leqslant 0$.

To bound the first two terms on the right-hand side of (B 33) observe that

$$M\,\partial F(\xi)/\partial\xi+\partial F(s)/\partial\tau < 0$$

implies that $$\{g(s)-F(s)f(s)\}^2 < \frac{\eta_0\tau^3}{\eta\xi^3}M\left(2-\frac{\xi^2}{\tau^2}\right)\{g(\xi)-F(\xi)f(\xi)\}^2.$$

Hence the sum of the first two terms on the right-hand side of (B 33) is greater than

$$\left\{\frac{\xi}{M\tau}+2-\frac{9\tau^2}{\xi^2}+\frac{6\tau^4}{\xi^4}-\frac{s\eta^2\tau^2}{\eta_0^2\xi^3}\left(2-\frac{\xi^2}{\tau^2}\right)\right\}\frac{Ms}{\tau^2\eta^3}\{g(\xi)-F(\xi)f(\xi)\}^2.$$

The quantity in the first brace is, since $\eta \leqslant \eta_0$, not less than

$$\frac{\xi}{M\tau}+1-\frac{7\tau^2}{\xi^2}+\frac{6\tau^4}{\xi^4}-2M\frac{\tau^3}{\xi^3}+\frac{M\tau}{\xi} = 1-\frac{\xi}{M\tau}+\frac{\tau^4}{\xi^4}\left(\frac{2\xi^5}{M\tau^5}-7\frac{\xi^2}{\tau^2}+6-2M\frac{\xi}{\tau}+M\frac{\xi^3}{\tau^3}\right)$$
$$\geqslant 1-\frac{\xi}{M\tau}+\frac{3\tau^4}{\xi^4}(2-M^2)(1-M^2),$$

because the entity in the parenthesis has a negative derivative with respect to ξ/τ and $\xi \leqslant M\tau$. Accordingly, the sum of the first two terms on the right-hand side of (B 33) is positive.

As far as the third term is concerned we start from

$$\frac{\mathrm{D}\{g(\xi)-F(\xi)f(\xi)\}}{g(\xi)-F(\xi)f(\xi)} = \frac{1}{3\eta}\left\{\frac{M}{\xi^3}\left(2-\frac{\xi^2}{\tau^2}\right)-\frac{1}{\tau^3}\right\}f(\xi)\{g(\xi)-F(\xi)f(\xi)\}$$
$$-M\eta-\frac{2}{3\tau^3}\int_0^{\xi}\{g(t)-F(\xi)f(t)\}f(t)\,\mathrm{d}t.$$

Now

$$\left\{\frac{1}{\xi^3}\left(2-\frac{\xi^2}{\tau^2}\right)-\frac{1}{M\tau^3}\right\}f(\xi)-\frac{3\eta^2}{g(\xi)-F(\xi)f(\xi)} = \left\{\frac{1}{\xi^3}\left(2-\frac{\xi^2}{\tau^2}\right)-\frac{1}{M\tau^3}\right\}f(\xi)-\frac{\eta^2}{\xi^2}\{f'(\xi)+\eta f(\xi)\}$$
$$\leqslant\left\{-\eta^4-\xi\eta^3+\frac{\xi^3}{M^2\tau^3}\left(\frac{\xi}{\tau}-M\right)+\frac{\xi^4}{\tau^4}\left(1-\frac{1}{M^2}\right)\right\}\frac{f(\xi)}{\xi^3}$$

on account of (A 9). Consequently, $\mathrm{D}\{g(\xi)-F(\xi)f(\xi)\} < 0$ and the last term of (B 29) supplies a positive contribution.

The fact that the left-hand side of (B 33) is positive where $\mathrm{D}\{F(\xi)+F(s)\} \leqslant 0$ leads to the deduction that $\tau^3\mathrm{D}\{F(\xi)+F(s)\}$ must be an increasing function of u there. One outcome is that once $\mathrm{D}\{F(\xi)+F(s)\}$ has assumed a positive value as u increases from $2/M$, it must remain positive thereafter.

The consequence is that Δ_0 has at most two zeros for $u > 2/M$. There can be two zeros only if Δ_0 is negative at $u = 2/M$, i.e. $F(s, 2s/M) > 0$. But then, according to the analysis of § B 7, there is no zero in $1 \leqslant Mu \leqslant 2$. Hence Δ_0 has at most two zeros on $u \geqslant 1/M$. It is perhaps worth noting that, if for $s = s_1$ there is a zero at $\tau = \tau_1$, there is also a zero at $\tau = \tau_1$ when $s = s_2$ where $s_2 = M\tau_1 - s_1$.

Whether there are two, one or no zeros is partially dictated by the behaviour of Δ_0 as $u \to \infty$. From (A 23), (A 25), (A 31), (A 51)–(A 54)

$$\Delta_0 \sim \tfrac{1}{3}(2s-1)\,\mathrm{e}^s/(s\,\mathrm{e}^s - \sinh s) \tag{B 34}$$

as $u \to \infty$. Thus, if $s > \frac{1}{2}$, Δ_0 has precisely one zero on $u \geqslant 1/M$. As s lessens and passes through $\frac{1}{2}$, a second zero appears. It originates at infinity and migrates towards the origin while the first zero marches in the opposite direction. Eventually the two zeros coalesce when s has fallen to a certain value, say s_0. A slight reduction in s causes the zeros to disappear and Δ_0 has no zeros on $u \geqslant 1/M$ for $s < s_0$.

The statement that a second zero joins the first as s passes through $\frac{1}{2}$ rather than the first zero leaving at infinity is justified by a more detailed consideration of the behaviour of Δ_0 at infinity. A more accurate assessment than (B 34) is based on the fact that (A 35) indicates that errors due to the terms in ξ_1 are exponentially small so that the main correction comes from applying (A 55) and (A 56) to the terms in s. Use of these equations reveals that the alteration to (B 34) is the addition of a positive multiple of

$$\{\tfrac{5}{4}(s-\tfrac{1}{2})^2\,\mathrm{e}^{2s} - \tfrac{1}{3}s^4 - \tfrac{5}{6}s^3 + \tfrac{5}{4}s^2 - \tfrac{5}{16}\,\mathrm{e}^{-2s}\}/\tau^2.$$

But

$$\mathrm{e}^{2s} > 1 + 2s + 2s^2 + \tfrac{4}{3}s^3 + \tfrac{2}{3}s^4,$$

$$\mathrm{e}^{-2s} < 1 - 2s + 2s^2 - \tfrac{4}{3}s^3 + \tfrac{2}{3}s^4$$

so that the additional quantity certainly exceeds $(\frac{1}{2}s^4 + \frac{5}{6}s^5)/\tau^2$ which is positive. Consequently, Δ_0 is always diminishing at infinity. In particular, Δ_0 can approach zero at infinity only if it has a zero at some earlier finite point.

There is thus a critical Strouhal number s_0 dividing the values of s into those where Δ_0 has at least one real zero on $u \geqslant 1/M$ and those where Δ_0 is non-vanishing.

This knowledge may be fused with that of § B 6 to give the conduct of Δ along the whole real axis. Assume that any zeros on $u \geqslant 1/M$ are circumvented by an indentation above. Then, as u moves from $-\infty$ to ∞, the phase of Δ starts at negative real values and makes a circuit of the origin in the clockwise sense of magnitude 3π when $s > \frac{1}{2}$. The magnitude is 4π when $\frac{1}{2} > s > s_0$ and 2π if $s < s_0$.

B 9. *The large semi-circle*

This section deals with the variations in phase of Δ for large $|u|$ and $0 \leqslant \mathrm{ph}\,u \leqslant \pi$. Suppose firstly that $0 \leqslant \mathrm{ph}\,u \leqslant \frac{1}{2}\pi - \delta$. Again replace ξ by $-\xi_1$ and use (A 4) to make the arguments of f and g into ξ_1. Then (A 23), (A 25), (A 31), (A 51)–(A 54) give

$$\Delta \sim 2u^2(1-\zeta^2)^{\frac{1}{2}}\,\mathrm{e}^s(2s-1)\,f(\zeta\tau, \tau),$$

where $\zeta = M - 1/u$. Consequently, for $|u| \to \infty$, $0 \leqslant \mathrm{ph}\, u \leqslant \frac{1}{2}\pi - \delta$

$$\Delta \sim 3u^2\tau\zeta(1-\zeta^2)^{\frac{1}{4}}(2s-1)\exp(s+\tfrac{1}{4}\pi\tau-\tfrac{2}{3}\tau z^{\frac{3}{2}}) \tag{B 35}$$

where z is expressed in terms of ζ by means of (A 16).

For $\frac{1}{2}\pi - \delta \leqslant \mathrm{ph}\, u \leqslant \frac{1}{2}\pi$, the metamorphosis in the asymptotic performance of f and g must be attended to and (A 27), (A 32) used. However, owing to the cancellation which takes place, it is necessary to choose the higher approximation (described after (A 33)) which includes B_0 but no other additional terms. This leads to, for example,

$$f' + (1-\zeta^2)^{\frac{1}{2}} f \sim 3\left[\tau\zeta(1-\zeta^2)^{\frac{1}{4}}\left\{1 - \frac{5}{48\tau z^{\frac{3}{2}}} - \frac{z^{\frac{1}{2}}}{\tau}B_0(z)\right\} + \tfrac{1}{2}\nu\right]$$
$$\times \exp(\tfrac{1}{4}\pi\tau - \tfrac{2}{3}\tau z^{\frac{3}{2}}) + \tfrac{3}{2}\nu\exp[-\tfrac{1}{4}\pi\tau + \tfrac{2}{3}\tau z^{\frac{3}{2}}],$$

$$g' + (1-\zeta^2)^{\frac{1}{2}} g \sim -\left[\tau\zeta(1-\zeta^2)^{\frac{1}{4}}\left\{1 - \frac{5}{48\tau z^{\frac{3}{2}}} - \frac{z^{\frac{1}{2}}}{\tau}B_0(z)\right\} + \tfrac{1}{2}\nu\right]$$
$$\times \exp(\tfrac{1}{4}\pi\tau - \tfrac{2}{3}\tau z^{\frac{3}{2}}) + \tfrac{1}{2}\nu\exp[-\tfrac{1}{4}\pi\tau + \tfrac{2}{3}\tau z^{\frac{3}{2}}],$$

where
$$\nu = \frac{\mathrm{d}}{\mathrm{d}\zeta}\frac{\zeta}{(1-\zeta^2)^{\frac{1}{4}}} = \frac{1-\frac{1}{2}\zeta^2}{(1-\zeta^2)^{\frac{5}{4}}}$$

and $B_0(z)$ is specified in (A 34). Following along these lines, we discover that

$$\Delta \sim 3u^2(2s-1)\left[\tau\zeta(1-\zeta^2)^{\frac{1}{4}}\left\{1 - \frac{5}{48\tau z^{\frac{3}{2}}} - \frac{z^{\frac{1}{2}}}{\tau}B_0(z)\right\} + \tfrac{1}{2}\nu\right]$$
$$\times \exp[s + \tfrac{1}{4}\pi\tau - \tfrac{2}{3}\tau z^{\frac{3}{2}}] - \tfrac{3}{2}u^2\nu\exp[-s - \tfrac{1}{4}\pi\tau + \tfrac{2}{3}\tau z^{\frac{3}{2}}] \tag{B 36}$$

as $|u| \to \infty$ with $\frac{1}{2}\pi - \delta \leqslant \mathrm{ph}\, u \leqslant \frac{1}{2}\pi$.

If $0 \leqslant \mathrm{ph}\, u \leqslant \frac{1}{2}\pi - \delta$, the difference between (B 36) and (B 35) contains only terms of smaller order than those retained in (B 35). Therefore, (B 36) may be safely employed, as far as our undertaking is concerned, for the whole interval $0 \leqslant \mathrm{ph}\, u \leqslant \frac{1}{2}\pi$.

On the interval $\frac{1}{2}\pi \leqslant \mathrm{ph}\, u \leqslant \pi$ do not alter ξ but put $-\tau'$ in place of τ. Replace $-\tau'$ by τ' in the second argument of f and g by virtue of (A 5). Since $-\frac{1}{2}\pi \leqslant \mathrm{ph}\, \tau' \leqslant 0$, the appropriate formulae are (A 29) and (A 33) with ζ still equal to $M - 1/u$. If $-\frac{1}{2}\pi + \delta \leqslant \mathrm{ph}\, \tau' \leqslant 0$, substitution of the asymptotic expressions reveals that

$$\Delta \sim -\tfrac{3}{2}u^2\nu\exp[\tfrac{1}{4}\pi\tau' - \tfrac{2}{3}\tau' z^{\frac{3}{2}} - s] = -\tfrac{3}{2}u^2\nu\exp[-\tfrac{1}{4}\pi\tau + \tfrac{2}{3}\tau z^{\frac{3}{2}} - s]. \tag{B 37}$$

The differences between (B 37) and (B 36) in this range are negligibly small and so (B 36) may also be employed for these values of $\mathrm{ph}\, u$.

Finally, for $-\frac{1}{2}\pi \leqslant \mathrm{ph}\, \tau' \leqslant -\frac{1}{2}\pi + \delta$, we elicit that (B 36) also holds. Thus (B 36) may be accepted as the relevant formula for the total interval $0 \leqslant \mathrm{ph}\, u \leqslant \pi$.

It is not by any means a trivial exercise to trace the variations in $\mathrm{ph}\, \Delta$ from (B 36) as the semicircle is encompassed. If we extract the factor $\frac{3}{2}u^2\nu\exp[-s-\frac{1}{4}\pi\tau+\frac{2}{3}\tau z^{\frac{3}{2}}]$ whose phase variation is 2π in the anti-clockwise sense, we are left with

$$(2s-1)\left[\frac{2\tau}{\nu}\zeta(1-\zeta^2)^{\frac{1}{4}}\left\{1 - \frac{5}{48\tau z^{\frac{3}{2}}} - \frac{z^{\frac{1}{2}}}{\tau}B_0(z)\right\} + 1\right]\exp[2s + \tfrac{1}{2}\pi\tau - \tfrac{4}{3}\tau z^{\frac{3}{2}}] - 1.$$

Since $|u|$ is large, functions of ζ can be expanded about $\zeta = M$ and only second orders kept to ensure that the approximation is consistent. When this is done, the expression reduces to

$$m(\tau) = (2s-1)(\alpha_1\tau+\beta_1)\exp\{2s-2s(1-M^2)^{\frac{1}{2}}+c\tau\}-1,$$

where $(1-\frac{1}{2}M^2)\alpha_1 = 2M(1-M^2)^{\frac{3}{2}}$,

$$(1-\tfrac{1}{2}M^2)\beta_1 = -1+3M^2-\tfrac{23}{12}M^4-(2-3M^2)(1-M^2)^{\frac{1}{2}}s-2M^2(1-M^2)s^2,$$

$$c = \arcsin M+M(1-M^2)^{\frac{1}{2}}.$$

On the real τ-axis the sign of the derivative of $m(\tau)$ is that of $\alpha_1+c(\alpha_1\tau+\beta_1)$. It therefore changes from negative to positive values as τ travels from $-\infty$ to ∞. If $s > \frac{1}{2}$, $m(\tau)$ is negative at $-\infty$ and positive at $+\infty$. Accordingly, $m(\tau)$ has exactly one real zero if $s > \frac{1}{2}$.

If $s < \frac{1}{2}$, $m(\tau)$ is negative at both $\pm\infty$ and so possesses either two real zeros or none. The elucidation is crucial to Δ and we wish to demonstrate that $m(\tau)$ has no real zero for $s < \frac{1}{2}$.

The maximum of $m(\tau)$ on the real axis is

$$(1-2s)(\alpha_1/c)\exp[2s-2s(1-M^2)^{\frac{1}{2}}-1-\beta_1 c/\alpha_1]-1.$$

Now $\sin 2\theta < 2\theta$ for $\theta > 0$ and so $c > 2M(1-M^2)^{\frac{1}{2}}$. Hence

$$\alpha_1/c \leqslant (1-M^2)/(1-\tfrac{1}{2}M^2) < 1$$

unless $M = 0$, which case can be excluded without loss. Also

$$(1-\tfrac{1}{2}M^2)\beta_1 = \tfrac{1}{12}M^4-(1-M^2)^2-2s(1-M^2)^{\frac{5}{2}}-M^2(1-M^2)^{\frac{1}{2}}\{s+(1-M^2)^{\frac{1}{2}}\}\{2s(1-M^2)^{\frac{1}{2}}-1\}$$
$$> -(1-M^2)^2\{1+2s(1-M^2)^{\frac{1}{2}}\}.$$

Moreover, $\theta \leqslant \tan\theta$ for $0 \leqslant \theta \leqslant \frac{1}{2}\pi$ so that $\arcsin M \leqslant M(1-M^2)^{\frac{1}{2}}$ and $c < 2M(1-M^2)^{-\frac{1}{2}}$. Hence

$$-\beta_1 c/\alpha_1 < 1+2s(1-M^2)^{\frac{1}{2}}.$$

In addition, $(1-2s)\,e^{2s}$ is a decreasing function of s and so does not exceed unity. It is therefore evident that the maximum value of $m(\tau)$ is negative for $s < \frac{1}{2}$, i.e. $m(\tau)$ has no real zeros when $s < \frac{1}{2}$.

Remembering the conclusions of §B 8 we see that Δ has the same number of non-real zeros between the large semi-circle and the real axis as $m(\tau)$ when $s > \frac{1}{2}$ or $s < s_0$ but one less if $s_0 < s < \frac{1}{2}$. To put it another way, Δ has an infinite sequence $u_1^+, u_2^+, \ldots$ of non-real zeros in the upper half plane for any value of s. In addition, as s decreases through s_0, an extra non-real zero u_0 appears in the upper half plane, starting from a point in the real axis when $u > 1/M$.

B 10. *The lower half-plane*

The unsymmetric nature of Δ prevents our being able to deduce the zeros of Δ in the lower half-plane from those in upper and there is, in fact, a significant difference. Fortunately, the treatment can be substantially simpler.

When u is in the lower half-plane take a complex conjugate of Δ. Owing to the series expansions (A 2) and (A 3) for f and g the effect is to replace u by u^* and change the signs of both v and w. Thereby, the problem is converted to discussing the zeros in the upper half-plane of Δ with the signs of v and w reversed. Let us denote this expression by Δ^-.

On $1 \leqslant u \leqslant 1/M$,

$$\frac{\Delta^-}{u^2\{f'(\xi)+\eta f(\xi)\}} = F(\xi)\{f'(s)-\eta_0 f(s)\}-\{g'(s)-\eta_0 f(s)\}. \tag{B 38}$$

By (B 6) and (B 10) the right-hand side of (B 38) is a decreasing function of ξ. It is positive for $\xi = s$ and therefore positive for $\xi < s$. Hence Δ^- is positive on $1 \leqslant u \leqslant 1/M$.

The argument of §§ B 2 and B 3 can be transferred virtually unchanged, the principal difference being that Δ_0^- has a positive imaginary part instead of a negative one.

For $u \leqslant -1/(1-M)$,

$$\frac{\Delta^-}{u^2\{f'(s)-\eta_0 f(s)\}} = g'(\xi)+\eta g(\xi) - F_1(s)\{f'(\xi)+\eta f(\xi)\}, \tag{B 39}$$

where

$$F_1(\xi) = \frac{g'(\xi)-\eta g(\xi)}{f'(\xi)-\eta f(\xi)}. \tag{B 40}$$

Since

$$\frac{\partial}{\partial \xi} F_1(\xi) = \frac{3\xi(2-\xi^2/\tau^2)}{\eta\{f'(\xi)-\eta f(\xi)\}^2}, \tag{B 41}$$

the right-hand side of (B 39) is a decreasing function of s. Being positive at $s = \xi$, it must be positive for $s < \xi$. Therefore Δ^- is positive on $u \leqslant -1/(1-M)$.

Pulling these last three paragraphs together we see that the phase of Δ^- swings from positive real values back to positive real values by making a clockwise circuit round the origin of 2π as u traverses the real axis in $u \leqslant 1/M$.

On $u \geqslant 1/M$, Δ^- has the same sign as Δ_0^- where

$$\Delta_0^- = -F_1(\xi) - F_1(s).$$

But $F_1(\xi)$ is an increasing function of ξ by (B 41) and so attains its largest value at $\xi = \tau$. Now

$$\frac{\mathrm{d}}{\mathrm{d}\tau} F_1(\tau) = \frac{2}{3\tau^3}\int_0^\tau \{g(t)-F_1(\tau)f(t)\}^2\,\mathrm{d}t$$

by (B 15) and (B 41). Thus $F_1(\tau)$ achieves its maximum at infinity. The asymptotic formulae for f and g, being uniformly valid, imply that this maximum is negative. Thus F_1 is always negative with the consequence that Δ_0^- is positive on the entire portion $u \geqslant 1/M$.

An investigation, similar to § B 9, provides the information that on the large semi-circle

$$\Delta^- \sim \tfrac{3}{2}u^2\nu \exp\left[s+\tfrac{1}{4}\pi\tau-\tfrac{2}{3}\tau z^{\frac{3}{2}}\right]$$
$$-3u^2(2s+1)\left[\zeta(1-\zeta^2)^{\frac{1}{2}}\left\{\tau+\frac{5}{48z^{\frac{3}{2}}}+z^{\frac{1}{2}}B_0(z)\right\}-\tfrac{1}{2}\nu\right]\exp\left[-s-\tfrac{1}{4}\pi\tau+\tfrac{2}{3}\tau z^{\frac{3}{2}}\right].$$

Accordingly, Δ^- has an infinite sequence of zeros in the upper half-plane, the same number as a function of similar character to $m(\tau)$.

We conclude that, in the lower half-plane Δ has no real zeros (except for the one at the origin) and an infinite sequence of non-real zeros $u_1^-, u_2^-, \ldots$.

B 11. *Summary*

It is convenient now to draw together the threads of the preceding argument and summarize the information it conveys about the zeros of Δ.

For all positive values of s, Δ has a simple zero at the origin.

With regard to other zeros, the upper half-plane will be dealt with first.

If $s > \frac{1}{2}$, Δ has a single (simple) real zero in $u \geqslant 1/M$ and an infinite sequence of non-real zeros $u_1^+, u_2^+, \ldots$ in the upper half-plane. There are no further zeros in this region.

If $\frac{1}{2} > s > s_0$, there is an additional real zero in $u \geqslant 1/M$ but no other change from $s > \frac{1}{2}$.

If $s < s_0$, there are no real zeros in $u \geqslant 1/M$ and the non-real zeros are supplemented by the extra zero u_0, which stems from the coalescence of the two real zeros.

In the lower half-plane, Δ has no real zeros (other than the origin) but, for all s, possesses an infinite sequence of non-real zeros $u_1^-, u_2^-, \ldots$.

It may seem surprising that the coalescence of the two real zeros in the upper half-plane does not produce a non-real zero in the lower half-plane. However, any zero generated below the real axis by this process would lie on a different Riemann surface from the lower half-plane.

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